MATH 2107-601 **EXAM #2 Study Guide** NAME: ____________________
(Print)

THURSDAY

NAME: KEY
(Signature)

November 16, 2006

[6:30 p.m. - 7:45 p.m.] STUDENT I. D.: ____________________

INSTRUCTIONS: There are 2 parts to this exam, the main part plus an optional **EXTRA CREDIT** part (on the last page). The main part is worth 100 points while the extra credit part is worth 12 points. But any score over 100 will be truncated to 100.

Clarity of exposition (including proper spelling and punctuation is an integral part of a correct solution to any problem. In particular, **DO NOT PUT EQUAL SIGNS BETWEEN THINGS THAT ARE NOT EQUAL.** But do put them where they belong.

It is necessary to show all your work.

NOTE: EXAM #2 will be given on **Tuesday, November 21, 2006**.

GOOD LUCK!

PLEASE SIGN THE FOLLOWING STATEMENT:
On my honor, I declare that the work that follows is entirely my own. With regard to all the questions on this exam, I have neither given nor received help from anyone [including myself, say, via any type of cheat sheet or device (e.g., cell phone)]. Nor have I used a calculator of any sort (i.e., regular or programmable).

NAME: ____________________
(Signature)

PLEASE DO NOT WRITE BELOW THIS LINE:

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1. a. ____	5. a. ____	**EXTRA-CREDIT:**
b. ____	b. ____	1. ____
2. a. ____	c. ____	2. ____
b. ____	d. ____	**TOTALS:**
3. ____	e. ____	Column 3: ____
4. a. ____	f. ____	Column 2: ____
b. ____	g. ____	Column 1: ____
c. ____	h. ____	**Grand Total:** ____
d. ____		**Grade:** ____
Total: ____	Total: ____	

#3. $(*)$ $\boxed{x^4 + x^3 y^3 + y^4 = 1 \quad \text{with } y=-1 \text{ when } x=1.}$

Assume $y = J(x)$ defines y as a differentiable function of x satisfying $(*)$. Then

$$1 = x^4 + x^3 \cdot [J(x)]^3 + [J(x)]^4 \quad \text{with } J(1) = -1.$$

$$\Downarrow$$

$$0 = 1' = \left(x^4 + x^3 \cdot [J(x)]^3 + [J(x)]^4\right)'$$

$$= (x^4)' + \left(x^3 \cdot [J(x)]^3\right)' + \left([J(x)]^4\right)'$$

$$= 4x^3 + x^3 \cdot \left([J(x)]^3\right)' + [J(x)]^3 \cdot (x^3)' + \left([J(x)]^4\right)'$$

$$= 4 \cdot x^3 + x^3 \cdot 3 \cdot [J(x)]^2 \cdot J'(x) + [J(x)]^3 \cdot 3x^2 + 4 \cdot [J(x)]^3 \cdot J'(x)$$

$$= 4 \cdot x^3 + 3 \cdot x^3 [J(x)]^2 \cdot J'(x) + 3 \cdot x^2 \cdot [J(x)]^3 + 4 \cdot [J(x)]^3 \cdot J'(x)$$

$$= 4x^3 + 3 \cdot x^2 \cdot [J(x)]^3 + \left(3 \cdot x^3 \cdot [J(x)]^2 + 4 \cdot [J(x)]^3\right) \cdot J'(x)$$

$$\Downarrow$$

$$\left(3 \cdot x^3 \cdot [J(x)]^2 + 4 \cdot [J(x)]^3\right) \cdot J'(x) = -\left(4 \cdot x^3 + 3 \cdot x^2 \cdot [J(x)]^3\right)$$

$$\Downarrow$$

$$J'(x) = \frac{-\left(4 \cdot x^3 + 3 \cdot x^2 \cdot [J(x)]^3\right)}{3 \cdot x^3 \cdot [J(x)]^2 + 4 \cdot [J(x)]^3}$$

$$\Downarrow$$

$$m = J'(1) = \frac{-\left(4 \cdot 1^3 + 3 \cdot 1^2 \cdot (-1)^3\right)}{3 \cdot 1^3 \cdot [-1]^2 + 4 \cdot [-1]^3} = \frac{-\left(4 \cdot 1 + 3 \cdot 1 \cdot (-1)\right)}{3 \cdot 1 \cdot 1 + 4 \cdot (-1)}$$

$$= -\frac{(4-3)}{3-4} = \frac{-1}{-1} = 1$$

$$\Downarrow$$

when $x = 1$

$$y = J(x) = m \cdot x + b = 1 \cdot x + b \implies b = J(x) - x = J(1) - 1 = -1 - 1 = -2$$

$$\Downarrow$$

$$y = m \cdot x + b = (+1)x + b = 1 \cdot x + -2 = 1 \cdot x - 2$$

Alternatively,

$$x^4 + x^3 \cdot y^3 + y^4 = 1$$

$$\Downarrow$$

$$0 = 1' = (x^4 + x^3 \cdot y^3 + y^4)' = (x^4)' + (x^3 \cdot y^3)' + (y^4)'$$

$$= 4 \cdot x^3 + x^3 \cdot (y^3)' + y^3 \cdot (x^3)' + 4y^3 \cdot y'$$

$$= 4 \cdot x^3 + x^3 \cdot 3 \cdot y^2 \cdot y' + y^3 \cdot 3 \cdot x^2 + 4 \cdot y^3 \cdot y'$$

$$= (4 \cdot x^3 + 3 \cdot x^2 \cdot y^3) + (3 \cdot x^3 \cdot y^2 + 4 \cdot y^3) \cdot y'$$

$$\Downarrow$$

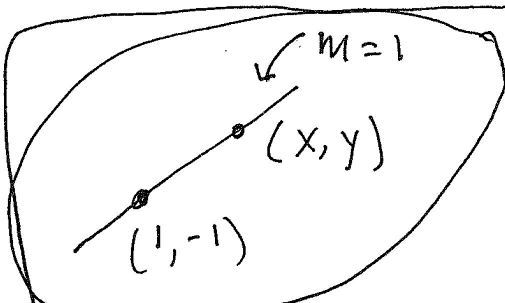
$$(3 \cdot x^3 \cdot y^2 + 4 \cdot y^3) \cdot y' = -(4 \cdot x^3 + 3 \cdot x^2 \cdot y^3)$$

$$\Downarrow$$

$$y' = \frac{-(4 \cdot x^3 + 3 \cdot x^2 \cdot y^3)}{3 \cdot x^3 \cdot y^2 + 4 \cdot y^3}$$

$$m = y'\Big|_{\substack{x=1,\\ y=-1}} = \frac{-(4 \cdot 1^3 + 3 \cdot 1^2 \cdot (-1)^3)}{3 \cdot 1^3 \cdot (-1)^2 + 4 \cdot (-1)^3}$$

$$= \frac{-(4 \cdot 1 + 3 \cdot 1 \cdot (-1))}{3 \cdot 1 \cdot 1 + 4 \cdot (-1)} = \frac{-(4-3)}{3-4} = \frac{-1}{-1} = 1$$



$$\Rightarrow 1 = m = \frac{y-(-1)}{x-1} = \frac{y+1}{x-1}$$

$$\Downarrow$$

$$y + 1 = 1 \cdot (x-1) = x - 1$$

$$\Downarrow$$

$$y = x - 1 - 1 = 1 \cdot x - 2$$

OR

$$y = m \cdot x + b = 1 \cdot x + b$$

$$\Downarrow$$

$$b = y - x\Big|_{\substack{x=1,\\ y=-1}} = -1 - 1 = -2$$

$$\Longrightarrow y = m \cdot x + b = 1 \cdot x + -2$$

$$= 1 \cdot x - 2$$

Alternatively,

$$x^4 + x^3 \cdot y^3 + y^4 = 1$$

$$\Downarrow$$

$$0 = \frac{d(1)}{dx} = \frac{d\left(x^4 + x^3 \cdot y^3 + y^4\right)}{dx} = \frac{d(x^4)}{dx} + \frac{d(x^3 \cdot y^3)}{dx} + \frac{d(y^4)}{dx}$$

$$= 4 \cdot x^3 + x^3 \cdot \frac{d(y^3)}{dx} + y^3 \cdot \frac{d(x^3)}{dx} + \frac{d(y^4)}{dx}$$

$$= 4 \cdot x^3 + x^3 \cdot \frac{d(y^3)}{dy} \cdot \frac{dy}{dx} + y^3 \cdot 3 \cdot x^2 + \frac{d(y^4)}{dy} \cdot \frac{dy}{dx}$$

$$= 4 \cdot x^3 + x^3 \cdot 3 \cdot y^2 \cdot \frac{dy}{dx} + 3 \cdot x^2 \cdot y^3 + 4 \cdot y^3 \cdot \frac{dy}{dx}$$

$$= \left(4 \cdot x^3 + 3 \cdot x^2 y^3\right) + \left(3 \cdot x^3 \cdot y^2 + 4 \cdot y^3\right) \cdot \frac{dy}{dx}$$

$$\Downarrow$$

$$\left(3 \cdot x^3 \cdot y^2 + 4 \cdot y^3\right) \cdot \frac{dy}{dx} = -\left(4 \cdot x^3 + 3 \cdot x^2 \cdot y^3\right)$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{-\left(4 \cdot x^3 + 3 \cdot x^2 y^3\right)}{3 \cdot x^3 \cdot y^2 + 4 \cdot y^3}$$

$$\Downarrow$$

$$m = \left.\frac{dy}{dx}\right|_{\substack{x=1\\ y=-1}} = \left.\frac{-\left(4 \cdot x^3 + 3 \cdot x^2 \cdot y^3\right)}{3 \cdot x^3 \cdot y^2 + 4 \cdot y^3}\right|_{\substack{x=1,\\ y=-1}}$$

$$= \frac{-\left(4 \cdot (+1)^3 + 3 \cdot 1^2 \cdot (-1)^3\right)}{3 \cdot 1^3 \cdot (-1)^2 + 4(-1)^3} = \frac{-\left(4 \cdot 1 + 3 \cdot 1 \cdot (-1)\right)}{3 \cdot 1 \cdot 1 + 4(-1)} = \frac{-(4-3)}{3-4} = \frac{-1}{-1} = 1$$

Then $y = m \cdot x + b = 1 \cdot x + b = x + b \Rightarrow b = y - x\Big|_{x=1,\, y=-1} = -1-1 = -2$

so $y = m \cdot x + b = 1 \cdot x + -2 = 1 \cdot x - 2$

4.a. $f(x) = \frac{x^3}{x^3+1} = \frac{x^3+1-1}{x^3+1} = \frac{x^3+1}{x^3+1} + \frac{-1}{x^3+1} = 1 + (-1)(x^3+1)^{-1}$

$\Downarrow$

$f'(x) = 0 + (-1)\cdot(-1)\cdot(x^3+1)^{-2}\cdot 3x^2 = \frac{3x^2}{(x^3+1)^2}$

$\boxed{\text{OR}}$ $f(x) = \frac{x^3}{x^3+1}$

$f'(x) = \frac{(x^3+1)\cdot 3x^2 - x^3\cdot(3x^2+0)}{(x^3+1)^2} = \frac{3x^5+3x^2-3x^5}{(x^3+1)^2} = \frac{3x^2}{(x^3+1)^2}$

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b. $f(x) = \left(\frac{x^4-1}{x^4+1}\right)^5 = [J(x)]^5 \Rightarrow J'(x) = 5\cdot[J(x)]^4\cdot J'(x)$

$f'(x) = 5\cdot\left(\frac{x^4-1}{x^4+1}\right)^4 \cdot \left[\frac{(x^4+1)\cdot 4x^3 - (x^4-1)\cdot 4x^3}{(x^4+1)^2}\right]$

$= 5\cdot\frac{(x^4-1)^4}{(x^4+1)^4} \cdot \left[\frac{4x^7+4x^3-4x^7+4x^3}{(x^4+1)^2}\right] = \frac{5\cdot(x^4-1)^4\cdot 8x^3}{(x^4+1)^6}$

$= \frac{40x^3\cdot(x^4-1)^4}{(x^4+1)^6}$

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c. $f(x) = \frac{\cos x}{1-\sin x} =: \frac{c}{1-s}$

$f'(x) = \left(\frac{c}{1-s}\right)' = \frac{(1-s)\cdot c' - c\cdot(1-s)'}{(1-s)^2} = \frac{(1-s)\cdot(-s) - c(-c)}{(1-s)^2}$

$= \frac{-s+s^2+c^2}{(1-s)^2} = \frac{s^2+c^2-s}{(1-s)^2} = \frac{1-s}{(1-s)^2} = \frac{1}{1-s} := \frac{1}{1-\sin x}$

---

d. $f(x) = \frac{1-\sin x}{\cos x} =: \frac{1-s}{c} = \frac{1}{c} - \frac{s}{c} := \sec x - \tan x$

$\Downarrow$

$f'(x) = \sec x\cdot\tan x - \sec^2 x$

$\boxed{\text{OR}}$ $f(x) = \left(\frac{1-s}{c}\right)' = \frac{c\cdot(1-s)' - (1-s)\cdot c'}{c^2} = \frac{c\cdot(0-c) - (1-s)(-s)}{c^2}$

$= \frac{c\cdot(-c) + (1-s)\cdot s}{c^2} = \frac{-c^2+s-s^2}{c^2} = \frac{s-(c^2+s^2)}{c^2} = \frac{s-1}{c^2}$

$= \frac{s}{c^2} - \frac{1}{c^2} = \frac{s}{c^2} - \frac{1^2}{c^2} = \frac{1}{c}\cdot\frac{s}{c} - \left(\frac{1}{c}\right)^2 := \sec x\cdot\tan x - \sec^2 x.$

5

5. a. $f(x) = \csc(x^6+1) = \csc[J(x)]$

$\Downarrow$

$f'(x) = -\csc[J(x)] \cdot \cot[J(x)] \cdot J'(x)$

$= -\csc(x^6+1) \cdot \cot(x^6+1) \cdot 6x^5$

---

b. $f(x) = e^{\tan[\ln(x^8+4)]} = e^{J(x)}$

$\Downarrow$

$f'(x) = e^{J(x)} \cdot J'(x)$

$= e^{\tan[\ln(x^8+4)]} \cdot \left(\sec^2[\ln(x^8+4)] \cdot \left(\left(\frac{1}{x^8+4} \right) \cdot 8x^7 \right) \right)$

$= e^{\tan[\ln(x^8+4)]} \cdot \sec^2[\ln(x^8+4)] \cdot \left(\frac{8x^7}{x^8+4} \right)$

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c. $f(x) = \sinh[\tan^{-1}(x^6)] = \sinh[J(x)]$

$\Downarrow$

$f'(x) = \cosh[J(x)] \cdot J'(x)$

$= \cosh[\tan^{-1}(x^6)] \cdot \left(\left[\frac{1}{1+(x^6)^2} \right] \cdot (6x^5) \right)$

$= \cosh[\tan^{-1}(x^6)] \cdot \left(\frac{6x^5}{1+x^{12}} \right)$

---

d. $f(x) = \cosh[\sin^{-1}(8^x)] = \cosh[J(x)] \cdot J'(x)$

$f'(x) = \sinh[\sin^{-1}(8^x)] \cdot \left(\left[\frac{1}{\sqrt{1-(8^x)^2}} \right] \cdot 8^x \cdot 1 \cdot \ln(8) \right)$

$= \sinh[\sin^{-1}(8^x)] \cdot \frac{8^x \cdot 1 \cdot \ln(8)}{\sqrt{1-8^{2x}}}$

5.e. $f(x) = 5^{(x^3+3x+2)} = a^g$

$\Downarrow$

$f'(x) = a^g \cdot g' \cdot \ln(a) = 5^{(x^3+3x+2)} \cdot (3x^2+3) \cdot \ln(5)$

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f. $f(x) = e^{x^{16}} \cdot \csc(x^4)$

$\Downarrow$

$f'(x) = e^{x^{16}} \cdot \left[-\csc(x^4) \cdot \cot(x^4) \cdot 4x^3\right]$
$+ \csc(x^4) \cdot e^{x^{16}} \cdot 16x^{15} \cdot 1$

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g. $f(x) = \sec(x^6) \cdot \cot(6^x)$

$\Downarrow$

$f'(x) = \sec(x^6) \cdot \left[-\csc^2(6^x) \cdot 6^x \cdot \ln(6)\right]$
$+ \cot(6^x) \cdot \left[\sec(x^6) \cdot \tan(x^6) \cdot 6x^5\right]$

---

h. $f(x) = \dfrac{(3x+4) \cdot \sqrt{5x+6}}{2x^2+3x+4} = \dfrac{F \cdot G^{1/2}}{H}$

$\Downarrow$

$\ln[f(x)] = \ln[F \cdot G^{1/2} \cdot H^{-1}] = \ln(F) + \frac{1}{2}\ln(G) - 1 \cdot \ln(H)$

$\Downarrow$

$\frac{1}{f(x)} \cdot f'(x) = \ln[f(x)]' = \left[\ln(F) + \frac{1}{2} \cdot \ln(G) - \ln(H)\right]'$

$= \frac{1}{F} \cdot F' + \frac{1}{2} \cdot \frac{1}{G} \cdot G' - \frac{1}{H} \cdot H'$

$\Downarrow$

$f'(x) = f(x)\left[\frac{F'}{F} + \frac{1}{2} \cdot \frac{G'}{G} - \frac{H'}{H}\right]$

$= \dfrac{(3x+4) \cdot \sqrt{5x+6}}{2x^2+3x+4} \cdot \left[\dfrac{3}{3x+4} + \dfrac{1}{2} \cdot \dfrac{5}{5x+6} - \dfrac{4x+3}{2x^2+3x+4}\right]$

7

$\boxed{\text{E.C.}}$

a. $f(x) = \left[\sin^4 x\right]^{\tan x} = \left[(\sin x)^4\right]^{\tan x} = F^G$

$\Downarrow$

$f'(x) = \left[F^G\right]' = G \cdot F^{G-1} \cdot F' + F^G \cdot G' \cdot \ln(F)$

$= (\tan x) \cdot \left[\sin^4(x)\right]^{[\tan(x) - 1]} \cdot \left(4 \cdot (\sin x)^3 \cdot \cos(x)\right)$

$+ \left[\sin^4 x\right]^{\tan(x)} \cdot \sec^2(x) \cdot \ln(\sin^4 x)$

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b. $f(x) = \log_7(7^{x^4}) + \log_7(x)$

$= x^4 \cdot \log_7(7) + \frac{\ln(x)}{\ln(7)}$

$= x^4 \cdot 1 + \frac{1}{\ln 7} \cdot \ln(x)$

$\Downarrow$

$f'(x) = 4x^3 + \frac{1}{\ln 7} \cdot \frac{1}{x}$

Note: $\log_7 x = y \iff 7^y = 7^{\log_7 x} = x$

$\Downarrow$

$y \cdot \ln 7 = \ln(7^y) = \ln x$

$\Downarrow$

$\log_7(x) = y = \frac{\ln x}{\ln 7}$

Note

$$y = F^G$$

$$\Downarrow$$

$$\ln y = \ln(F^G) = G \cdot \ln F$$

$$\Downarrow$$

$$\frac{1}{y} \cdot y' = (\ln y)' = (G \cdot \ln F)' = G \cdot (\ln F)' + (\ln F) \cdot G'$$

$$= G \cdot \frac{1}{F} \cdot F' + G' \cdot \ln F$$

$$= G \cdot F^{-1} \cdot F' + G' \cdot \ln F$$

$$\Downarrow$$

$$(F^G)' = y' = y \cdot [G \cdot F^{-1} \cdot F' + G' \cdot \ln F]$$

$$= F^G \cdot [G \cdot F^{-1} \cdot F' + G' \cdot \ln F]$$

$$= F^G \cdot G \cdot F^{-1} \cdot F' + F^G \cdot G' \cdot \ln F$$

$$= G \cdot F^G \cdot F^{-1} \cdot F' + F^G \cdot G' \cdot \ln F$$

$$= G \cdot F^{(G-1)} \cdot F' + F^G \cdot G' \cdot \ln F$$

In particular, for $G = c =$ constant,

$(F^c)' = c \cdot F^{c-1} \cdot F'$ since $c' = 0$

while, for $F = a =$ constant & $G = g$,

$(a^g)' = a^g \cdot g' \cdot \ln(a)$ since $a' = 0$