Math 2107

$$a^2 - b^2 = (a-b)\cdot(a+b) = (a-b)\cdot\underbrace{(a^1\cdot b^0 + a^0\cdot b^1)}_{2\text{ terms}}$$

List the a's in descending powers.

$$a^3 - b^3 = (a-b)\cdot\underbrace{(a^2\cdot b^0 + a^1\cdot b^1 + a^0 b^2)}_{3\text{ terms}}$$

List the b's in ascending powers.

$$a^4 - b^4 = (a-b)\cdot\underbrace{(a^3\cdot b^0 + a^2\cdot b^1 + a^1\cdot b^2 + a^0\cdot b^3)}_{4\text{ terms}}$$

All sums $= 4 = 5-1$

$$a^5 - b^5 = (a-b)\cdot\underbrace{(a^4\cdot b^0 + a^3\cdot b^1 + a^2\cdot b^2 + a^1\cdot b^3 + a^0\cdot b^4)}_{5\text{ terms}}$$

$\vdots$

$$a^n - b^n = (a-b)\cdot\underbrace{(a^{n-1}\cdot b^0 + a^{n-2}\cdot b^1 + \cdots + a^1 b^{n-2} + a^0\cdot b^{n-1})}_{(n-1)+1 = n\text{ terms}}$$

$\boxed{\text{Check for } n=5}$:

$$\begin{array}{l} a^4 + a^3 b + a^2 b^2 + a b^3 + b^4 \\ \underline{a - b \qquad\qquad\qquad\qquad\qquad} \\ a^5 + a^4 b + a^3 b^2 + a^2 b^3 + a b^4 \\ \underline{\quad\; - a^4 b - a^3 b^2 - a^2 b^3 - a b^4 - b^5} \\ a^5 \qquad\qquad\qquad\qquad\qquad\qquad\;\; - b^5 \end{array}$$

Claim: $f(x) = x^5 \Rightarrow f'(x) = 5x^4$

Proof: By definition,

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$a = x+h$
$b = x$
$\Downarrow$
$a - b = h$

Now

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^5 - x^5}{h} = \frac{a^5 - b^5}{h}$$

$$= \frac{(a-b)\cdot(a^4 + a^3 b + a^2 \cdot b^2 + a \cdot b^3 + b^4)}{h}$$

$$= \frac{h}{h} \cdot (a^4 + a^3 \cdot b + a^2 \cdot b^2 + a \cdot b^3 + b^4)$$

$$= a^4 + a^3 b + a^2 \cdot b^2 + a \cdot b^3 + b^4$$

$$= (x+h)^4 + (x+h)^3 \cdot x + (x+h)^2 \cdot x^2 + (x+h) \cdot x^3 + x^4$$

$$\therefore f'(x) = \lim_{h \to 0} \left[(x+h)^4 + (x+h)^3 \cdot x + (x+h)^2 \cdot x^2 + (x+h) \cdot x^3 + x^4 \right]$$

$$= x^4 + x^3 \cdot x + x^2 \cdot x^2 + x \cdot x^3 + x^4$$

$$= x^4 + x^4 + x^4 + x^4 + x^4$$

$$= 5 \cdot x^4$$

Claim: $f(x) = x^n \Rightarrow f'(x) = n \cdot x^{n-1} \quad \forall$ integer $n \geq 0$.

Proof: We already checked this if $n = 0, 1, 2, 3, 4$ or 5.

For any integer $n \geq 5$

$$a = x+h,\quad b = x \Rightarrow \boxed{a - b = h}$$

$$\frac{(x+h)^n - x^n}{h} = \frac{a^n - b^n}{h} = \frac{(a-b)\cdot(a^{n-1}\cdot b^0 + a^{n-2}\cdot b^1 + a^{n-3}\cdot b^2 + \dots + a^2\cdot b^{n-3} + a^1\cdot b^{n-2} + a^0\cdot b^{n-1})}{h}$$

$$= \frac{h}{h}\cdot\left(a^{n-1}\cdot b^0 + a^{n-2}\cdot b^1 + a^{n-3}\cdot b^2 + \dots + a^2\cdot b^{n-3} + a^1\cdot b^{n-2} + a^0\cdot b^{n-1}\right)$$

$$= \underbrace{a^{n-1}\cdot b^0 + a^{n-2}\cdot b^1 + a^{n-3}\cdot b^2 + \dots + a^2\cdot b^{n-3} + a^1\cdot b^{n-2} + a^0\cdot b^{n-1}}_{(n-1)+1\,=\,n \text{ terms}}$$

$$= (x+h)^{n-1} + (x+h)^{n-2}\cdot x + (x+h)^{n-3}\cdot x^2 + \dots + (x+h)^2\cdot x^{n-3} + (x+h)\cdot x^{n-2} + x^{n-1}$$

$$\therefore f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\left[(x+h)^{n-1} + (x+h)^{n-2}\cdot x + (x+h)^{n-3}\cdot x^2 + \dots + (x+h)^2\cdot x^{n-3} + (x+h)\cdot x^{n-2} + x^{n-1}\right]$$

$$= x^{n-1} + x^{n-2}\cdot x + x^{n-3}\cdot x^2 + \dots + x^2\cdot x^{n-3} + x\cdot x^{n-2} + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + x^{n-1} + \dots + x^{n-1} + x^{n-1} + x^{n-1} = n\cdot x^{n-1}$$

Alternate Proof that
$f(x) = x^5 \implies f'(x) = 5x^4$

Recall Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$\boxed{\text{etc}}$

$$(x+h)^5 = 1 \cdot x^5 \cdot h^0 + 5 \cdot x^4 \cdot h^1 + 10x^3 h^2 + 10 \cdot x^2 h^3 + 5 \cdot x^1 h^4 + 1 \cdot x^0 h^5$$

$$= x^5 + 5 \cdot x^4 \cdot h + 10 \cdot x^3 \cdot h^2 + 10 \cdot x^2 \cdot h^3 + 5 \cdot x \cdot h^4 + h^5$$

Hence

$$\frac{(x+h)^5 - x^5}{h} = \frac{x^5 + 5 \cdot x^4 h + 10 \cdot x^3 \cdot h^2 + 10 \cdot x^2 \cdot h^3 + 5 \cdot x \cdot h^4 + h^5 - x^5}{h}$$

$$= \frac{5 \cdot x^4 \cdot h + 10 \cdot x^3 \cdot h^2 + 10 \cdot x^2 \cdot h^3 + 5 \cdot x \cdot h^4 + h^5}{h}$$

$$= \left(5x^4 + 10 \cdot x^3 \cdot h + 10 \cdot x^2 \cdot h^2 + 5 \cdot x \cdot h^3 + h^4\right) \cdot \frac{h}{h}$$

$$= 5 \cdot x^4 + 10 \cdot x^3 \cdot h + 10 \cdot x^2 \cdot h^2 + 5 \cdot x \cdot h^3 + h^4$$

$$\therefore f'(x) = \underset{h \to 0}{L} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{L} \frac{(x+h)^5 - x^5}{h}$$

$$= \underset{h \to 0}{L} \left(5 \cdot x^4 + 10 \cdot x^3 h + 10 \cdot x^2 \cdot h^2 + 5 \cdot x \cdot h^3 + h^4\right)$$

$$= 5 \cdot x^4 + 10x^3 \cdot 0 + 10x^2 \cdot 0^2 + 5x \cdot 0^3 + 0^4 = 5 \cdot x^4.$$

Claim: $f(x) = \sqrt{x} = x^{1/2} \implies f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

Proof: $f(x) = x^{1/2}$

$$\Downarrow$$

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^{1/2} - x^{1/2}}{h}$$

Now

$$\frac{(x+h)^{1/2} - x^{1/2}}{h} = \frac{\sqrt{a} - \sqrt{b}}{h}$$

$\boxed{a = x+h, \quad b = x} \implies \boxed{a - b = h}$

$$= \frac{\sqrt{a} - \sqrt{b}}{h} \cdot \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$$

"Rationalize the Numerator"

$$= \frac{c-d}{h} \cdot \frac{c+d}{c+d} = \frac{c^2 - d^2}{h \cdot (c+d)} = \frac{a-b}{h \cdot (c+d)}$$

$\boxed{c = \sqrt{a}, \quad d = \sqrt{b}}$

$$\Downarrow$$

$\boxed{c^2 = a \ \& \ d^2 = b} \implies \boxed{c^2 - d^2 = a - b = h}$

$$= \frac{h}{h \cdot (c+d)}$$

$$= \frac{h}{h} \cdot \frac{1}{c+d}$$

$$= \frac{1}{c+d} = \frac{1}{\sqrt{a} + \sqrt{b}}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

Hence,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^{1/2} - x^{1/2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Claim: $f(x) = x^{1/3} \implies f'(x) = \frac{1}{3} x^{-2/3}$

Proof: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h}$

Now

$$\frac{(x+h)^{1/3} - x^{1/3}}{h} = \frac{a^{1/3} - b^{1/3}}{h} \qquad \boxed{\begin{matrix} a = x+h \\ b = x \end{matrix}} \implies \boxed{a - b = h}$$

We want ? so that

$$= \frac{c-d}{h} = \frac{c-d}{h} \cdot \frac{?}{?} = \frac{c^3 - d^3}{h \cdot ?} = \frac{h}{h} \cdot \frac{1}{?} = \frac{1}{?}$$

$$\boxed{\begin{matrix} c = a^{1/3} \\ d = b^{1/3} \end{matrix}} \Downarrow \boxed{\begin{matrix} c^3 = a \\ \& \ d^3 = b \end{matrix}} \Downarrow \boxed{c^3 - d^3 = a - b = h}$$

Now $c^3 - d^3 = (c-d) \cdot ? = (c-d) \cdot (c^2 + c d + d^2)$

because

$$\begin{array}{r} c^2 + c\cdot d + d^2 \\ c - d \\ \hline c^3 + c^2 d + c d^2 \qquad \\ - c^2 d - c d^2 - d^3 \\ \hline c^3 \qquad\qquad\qquad - d^3 \end{array}$$

Hence if we take $? = c^2 + cd + d^2$ everything will turn out as planned.

Specifically,

$$\frac{(x+h)^{1/3}-x^{1/3}}{h} = \frac{a^{1/3}-b^{1/3}}{h}$$

$a = x+h$
$b = x$
$\Rightarrow \boxed{a-b=h}$

$$= \frac{c-d}{h} = \frac{c-d}{h}\cdot\frac{?}{?}$$

$c = a^{1/3}$
$d = b^{1/3}$
$\Downarrow$
$c^3 = a$
& $d^3 = b$
$\Downarrow$
$\boxed{c^3-d^3 = a-b = h}$

"Rationalize the Numerator"

$$\left(= \frac{c^3-d^3}{h\cdot ?} = \frac{h}{h\cdot ?} = \frac{h}{h}\cdot\frac{1}{?} = \frac{1}{?}\right)$$

$$= \frac{c-d}{h}\cdot\frac{?}{?} = \frac{c-d}{h}\cdot\frac{c^2+cd+d^2}{c^2+c\cdot d+d^2}$$

$$= \frac{(c-d)\cdot(c^2+c\cdot d+d^2)}{h\cdot(c^2+c\cdot d+d^2)} = \frac{c^3-d^3}{h\cdot(c^2+c\cdot d+d^2)}$$

$$= \frac{h}{h\cdot(c^2+c\cdot d+d^2)} = \frac{h}{h}\cdot\frac{1}{c^2+c\cdot d+d^2}$$

$$= \frac{1}{c^2+c\cdot d+d^2} = \frac{1}{(a^{1/3})^2+a^{1/3}\cdot b^{1/3}+(b^{1/3})^2}$$

$$= \frac{1}{a^{2/3}+a^{1/3}\cdot b^{1/3}+b^{2/3}} = \frac{1}{(x+h)^{2/3}+(x+h)^{1/3}\cdot x^{1/3}+x^{2/3}}$$

$$\therefore f'(x) = \lim_{h\to 0}\frac{1}{(x+h)^{2/3}+(x+h)^{1/3}\cdot x^{1/3}+x^{2/3}} = \frac{1}{x^{2/3}+x^{1/3}\cdot x^{1/3}+x^{2/3}}$$

$$= \frac{1}{3\cdot x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

Claim: $f(x) = x^{1/4} \Rightarrow f'(x) = \frac{1}{4} x^{-\frac{3}{4}}$

Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^{1/4} - x^{1/4}}{h}$

Now

$$\frac{(x+h)^{1/4} - x^{1/4}}{h} = \frac{a^{1/4} - b^{1/4}}{h} \qquad \boxed{\begin{aligned} a &= x+h \\ b &= x \end{aligned}} \Rightarrow \boxed{a - b = h}$$

$$\boxed{\begin{aligned} c &= a^{1/4} \\ d &= b^{1/4} \end{aligned}} \Downarrow \boxed{c^4 = a \ \& \ d^4 = b} \Downarrow \boxed{c^4 - d^4 = a - b = h}$$

$$= \frac{c-d}{h} = \frac{c-d}{h} \cdot \frac{?}{?}$$

$$\left(= \frac{c^4 - d^4}{h \cdot ?} = \frac{h}{h \cdot ?} = \frac{h}{h} \cdot \frac{1}{?} = \frac{1}{?} \right)$$

$$= \frac{c-d}{h} \cdot \frac{?}{?} = \frac{c-d}{h} \cdot \frac{c^3 + c^2 d + c d^2 + d^3}{c^3 + c^2 \cdot d + c \cdot d^2 + d^3}$$

$$= \frac{(c-d) \cdot (c^3 + c^2 d + c \cdot d^2 + d^3)}{h \cdot (c^3 + c^2 \cdot d + c \cdot d^2 + d^3)}$$

$$= \frac{c^4 - d^4}{h \cdot (c^3 + c^2 d + c d^2 + d^3)}$$

$$= \frac{h}{h \cdot (c^3 + c^2 d + c \cdot d^2 + d^3)} = \frac{1}{c^3 + c^2 d + c d^2 + d^3}$$

$$= \frac{1}{(a^{1/4})^3 + (a^{1/4})^2 b^{1/4} + a^{1/4} (b^{1/4})^2 + (b^{1/4})^3}$$

$$= \frac{1}{a^{3/4} + a^{2/4} b^{1/4} + a^{1/4} b^{2/4} + b^{3/4}}$$

$$= \frac{1}{(x+h)^{3/4} + (x+h)^{2/4} \cdot x^{1/4} + (x+h)^{1/4} \cdot x^{2/4} + x^{3/4}}$$

$$\therefore f'(x) = \frac{1}{x^{3/4} + x^{2/4} \cdot x^{1/4} + x^{1/4} \cdot x^{2/4} + x^{3/4}} = \frac{1}{4 x^{3/4}}$$

Claim: $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -1 \cdot x^{-2} = \frac{-1}{x^2}$

Proof:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Now

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \frac{\frac{1}{x+h} \cdot \frac{x}{x} - \frac{x+h}{x+h} \cdot \frac{1}{x}}{h} \cdot \frac{\frac{1}{h}}{\frac{1}{h}}$$

$$= \frac{x - (x+h)}{(x+h) \cdot x} \cdot \frac{1}{h}$$

$$= \frac{x - x - h}{(x+h) \cdot x \cdot h} = \frac{-h}{(x+h) \cdot x \cdot h}$$

$$= \frac{-1}{(x+h) \cdot x} \cdot \frac{h}{h} = \frac{-1}{(x+h) \cdot x}$$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{-1}{(x+h) \cdot x} = \frac{-1}{(x+0) \cdot x} = -\frac{1}{x \cdot x}$$

$$= -\frac{1}{x^2} = -1 \cdot \frac{1}{x^2} = -1 \cdot x^{-2}.$$

Claim: $f(x) = m \cdot x + b$ for constants m & b

$$\Downarrow$$

$$f'(x) = m$$

Proof: This is obvious geometrically because $f'(x)$ = the slope of the straight line tangent to the graph of f at the point $(x, f(x))$.

But the graph of f is a straight line of slope m & having b as the y-value of the y-intercept $(= (0,b))$; and the tangent to a straight line at any point on it coincides with (i.e., is [identical to]) that straight line.

Analytically,

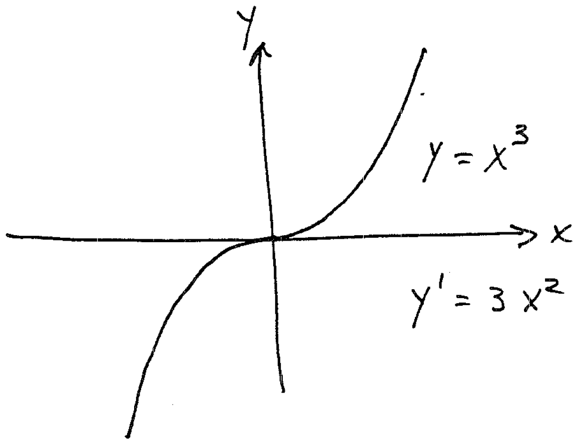
$$f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{m\cdot(x+h) + b - [mx+b]}{h}$$

$$= \lim_{h\to 0} \frac{m\cdot x + m\cdot h + b - m\cdot x - b}{h}$$

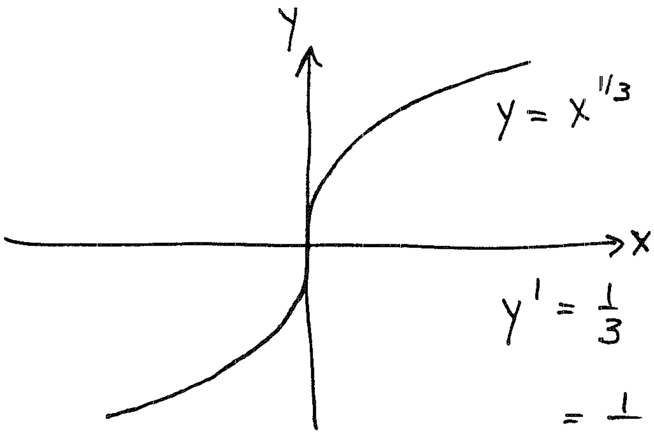
$$= \lim_{h\to 0} \frac{m\cdot h}{h} = \lim_{h\to 0} m\cdot\frac{h}{h} = \lim_{h\to 0} m = m.$$

Boundary Restrictions



$y = x^3$

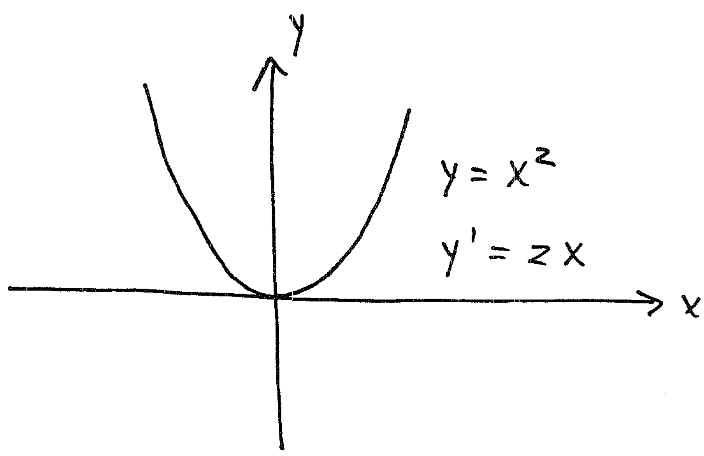
$y' = 3x^2 \geq 0$



$y = x^{1/3}$

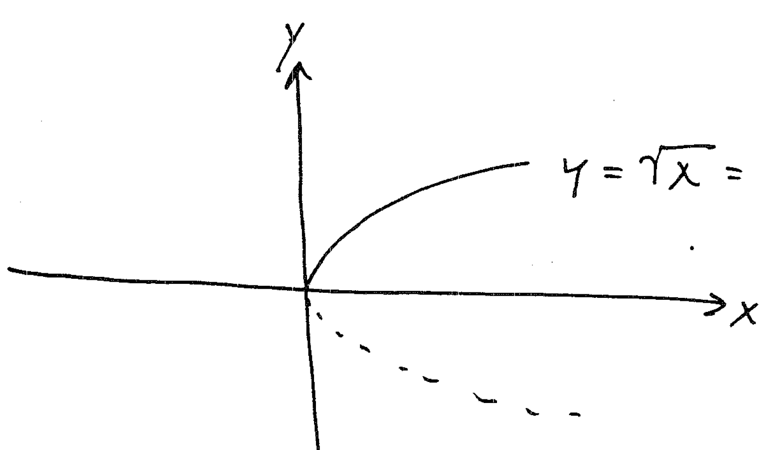
$y' = \frac{1}{3} x^{-1/3}$, $x \neq 0$

$= \frac{1}{3x^{1/3}}$, $x \neq 0$



$y = x^2$

$y' = 2x$



$y = \sqrt{x} = x^{1/2}$

$y' = \frac{1}{2} x^{-1/2}$, $x \neq 0$

$= \frac{1}{2\sqrt{x}}$, $x \neq 0$