MATH 2107-601 **EXAM #1 Study Guide** NAME: KEY
(Print)

TUESDAY

OCTOBER 8, 2002 NAME: ____________________
(Signature)

[7:30 p.m. - 8:45 p.m.] STUDENT I. D.: ____________________

INSTRUCTIONS: There are 2 parts to this exam, the main part plus an optional extra credit part (on the last page). The main part is worth 100 points while the extra credit part is worth 10 points - but any score over 100 will be truncated to 100.

Clarity of exposition is an integral part of a correct solution to any problem.

Good Luck.

PLEASE SIGN THE FOLLOWING STATEMENT:
On my honor, I declare that the work that follows is entirely my own. With regard to all the questions on this exam, I have neither given nor received help from anyone [including myself, say, via any type of cheat sheet or device (e.g., cell phone)]. Nor have I used a calculator of any sort (i.e., regular or programmable).

NAME: ____________________
(Signature)

PLEASE DO NOT WRITE BELOW THIS LINE:

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20 { 1. (16) a. 3	5. 10	**EXTRA-CREDIT:**
b. 4	6. a. 5	1. 6
c. 6	b. 5	2. 6
d. 3	c. 5	**TOTALS:**
2. (4) a. 1	d. 5	Column 3: 12
b. 3	e. 5	Column 2: 65
15 { 3. (11) a.(i) 2	f. 5	Column 1: 35
(ii) 2	g. 5	**Grade Total:** 112 > 100
b.(i) 5	h. 5	: A+
(ii) 2	i. 5	
4. 4	j. 5	
Total: 35	k. 5	
	Total: 65	

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1. (a) By definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

[3 points]

(b) Geometrically, $f'(x)$ is equal to the slope of

[4 points] the straight line tangent to the graph of f at the point $(x, f(x))$

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and that $a \in \mathbb{R}$. Then, by definition,

$$\lim_{x \to a} f(x) = L$$

if and only if for any given (error bound) $\varepsilon > 0$

[6 points] there exists a (deviation bound) $\delta > 0$ so that $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

(d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and that $a \in \mathbb{R}$. Then, by definition, f is <u>continuous at</u> a if and only if

[3 points] (1) f is defined at a,

(2) $\lim_{x \to a} f(x)$ exists,

and (3) $\lim_{x \to a} f(x) = f(a)$

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function. What characteristic of the graph of f enables you to tell at a glance

(a) that f is everywhere continuous?

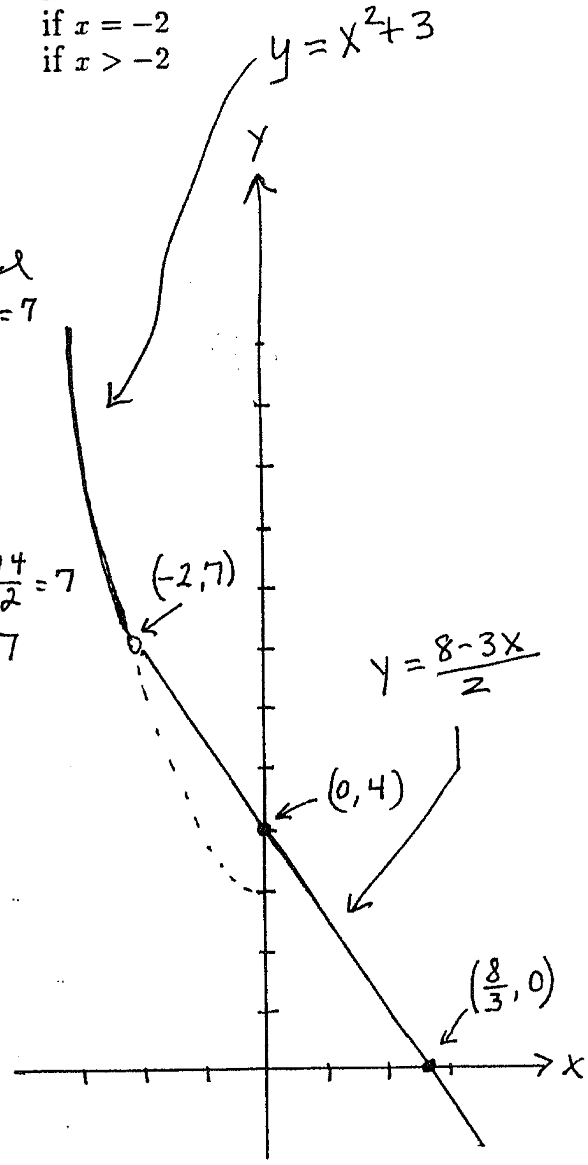
[1 point] The graph of f is unbroken (i.e., it has no jumps or breaks in it).

(b) that f is everywhere differentiable?

[3 points] The graph of f is unbroken and, additionally, has no corners, vertical tangents, or wild oscillations.

3.a. [11 points] Let

$$f(x) = \begin{cases} x^2+3 & , \quad \text{if } x < -2 \\ 0 & , \quad \text{if } x = -2 \\ 4 - \frac{3}{2}x & , \quad \text{if } x > -2 \end{cases}$$



(i) Does $\lim_{x\to -2} f(x)$ exist?

✓ Yes or ___ No

$\lim_{x\to-2^-} f(x) = 7 = \lim_{x\to-2^+} f(x) \Rightarrow \lim_{x\to-2} f(x)$ exists and $\lim_{x\to-2} f(x) = 7$

Explain Why or Why Not:

$$\lim_{x\to-2^-} f(x) = \lim_{\substack{x\to-2 \\ x<-2}} x^2+3 = (-2)^2+3 = 4+3 = 7$$

$$\lim_{x\to-2^+} = \lim_{\substack{x\to-2 \\ x>-2}} \frac{8-3x}{2} = \frac{8-3(-2)}{2} = \frac{8+6}{2} = \frac{14}{2} = 7$$

so $\lim_{x\to-2^-} f(x) = 7 = \lim_{x\to-2^+} f(x) \Rightarrow \lim_{x\to-2} f(x) = 7$

(ii) Is $f : \mathbb{R} \to \mathbb{R}$ continuous at $x = -2$?

___ Yes or ✓ No

Explain Why or Why Not:

$$\lim_{x\to-2} f(x) = 7 \neq 0 = f(-2)$$

i.e., $\lim_{x\to-2} f(x) \neq f(-2)$.

b.(i) With reference to the graph below, answer each of the following questions about the function $f : [-3,2] \to \mathbb{R}$.

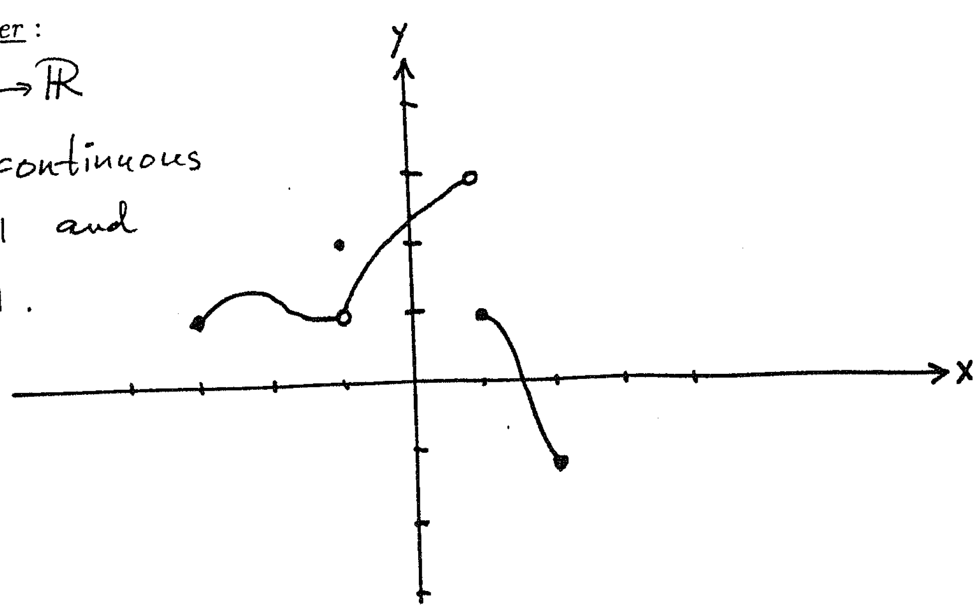
$\lim_{x\to-1^-} f(x) = 1$ $\lim_{x\to-1^+} f(x) = 1$

$\lim_{x\to-1} f(x) = 1$ $\lim_{x\to1^-} f(x) = 3$ $f(-1) = 2$

(ii) List all the points $x \in [-3,2]$ at which $f : [-3,2] \to \mathbb{R}$ is not continuous.

Answer :

$f : [-3,2] \to \mathbb{R}$ is not continuous at $x = -1$ and at $x = 1$.



4. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a point $a \in \mathbb{R}$ such that f is continuous at a but $f'(a)$ does <u>not</u> exist.

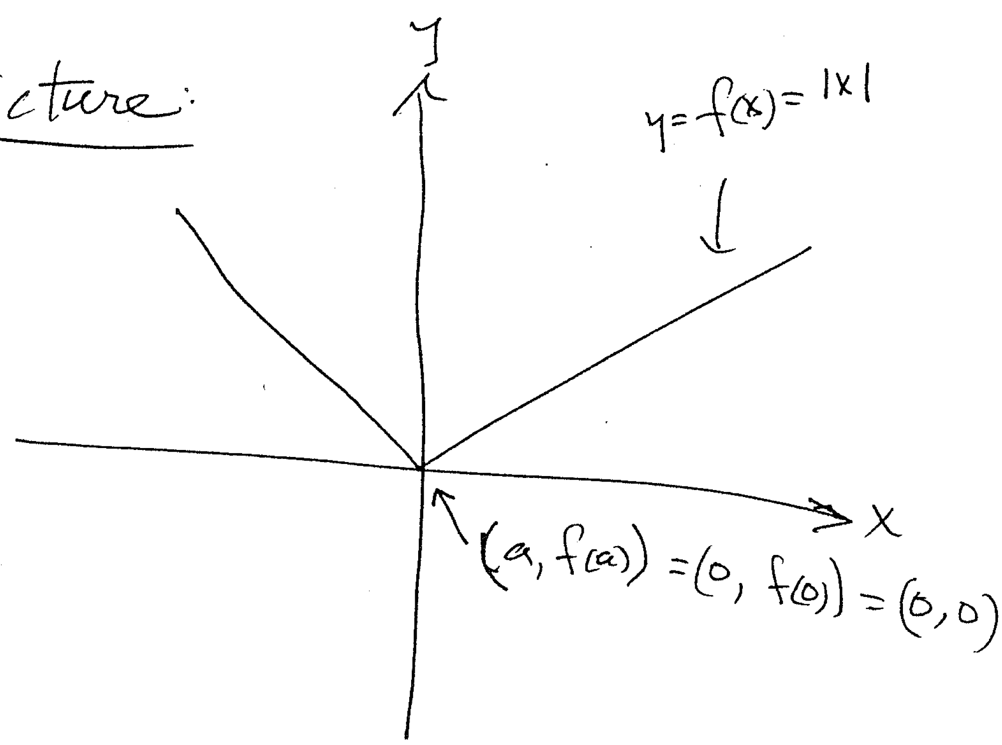
[To get credit, you must give at least some indication of why f is continuous at a and why $f'(a)$ does not exist.]

[4 points] Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) := |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

for all $x \in \mathbb{R}$, and let $a := 0 \in \mathbb{R}$

Picture:



Then, $f : \mathbb{R} \to \mathbb{R}$ is everywhere continuous since its graph is unbroken. Yet $f'(0)$ does not exist because, were it to exist then $f'(0)$ would equal the slope of the straight line tangent to the graph of f at $(0, f(0)) = (0,0)$. But the graph of f has a corner at $(0, f(0))$ & hence no tangent there!

5. Compute $f'(x)$ directly from the definition in case

$$f(x) = \frac{1}{x-5}\,.$$

NOTE: No credit will be given for just the answer.

[10 points]

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h-5} - \frac{1}{x-5}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1\cdot(x-5) - 1\cdot(x+h-5)}{(x+h-5)(x-5)}}{h} \cdot \frac{\frac{1}{h}}{\frac{1}{h}}$$

$$= \lim_{h\to 0} \frac{x-5-x-h+5}{(x+h-5)(x-5)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-h}{(x+h-5)\cdot(x-5)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{(-1)}{(x+h-5)\cdot(x-5)} \cdot \frac{h}{h}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-5)\cdot(x-5)}$$

$$= \frac{-1}{(x+0-5)\cdot(x-5)} = \frac{-1}{(x-5)^2}$$

$$= (-1)(x-5)^{-2}$$

6. Evaluate the following limits. [**No work, no credit.**]

[55 points: 5 each]

Note: By synthetic division:

$$\begin{array}{r|rrr} 2 & 1 & 1 & -6 \\ & & 2 & 6 \\ \hline & 1 & 3 & \boxed{0} \end{array} \Rightarrow x^2+x-6 = (x-2)(x+3)$$

Also

$$\begin{array}{r|rrr} 2 & 1 & -7 & 10 \\ & & 2 & -10 \\ \hline & 1 & -5 & \boxed{0} \end{array} \Rightarrow x^2-7x+10 = (x-2)(x-5)$$

(a) $\lim_{x\to 2} \dfrac{x^2+x-6}{x^2-7x+10} = \lim_{x\to 2} \dfrac{(x+3)\cdot(x-2)}{(x-5)\cdot(x-2)} = \lim_{x\to 2} \dfrac{x+3}{x-5}$

$= \dfrac{\lim_{x\to 2}(x+3)}{\lim_{x\to 2}(x-5)} = \dfrac{2+3}{2-5} = \dfrac{5}{-3} = -\dfrac{5}{3}$

(b) $\lim_{x\to 0}\left(\sqrt[3]{x-8} - \sqrt[3]{x-1}\right) = \lim_{x\to 0}\left(\sqrt[3]{x-8}\right) - \lim_{x\to 0}\left(\sqrt[3]{x-1}\right)$

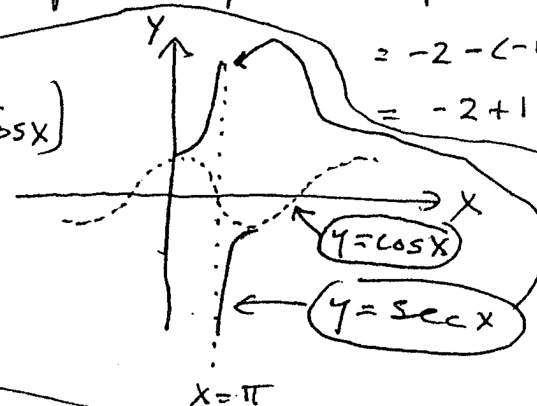
$= \sqrt[3]{0-8} - \sqrt[3]{0-1} = \sqrt[3]{-8} - \sqrt[3]{-1}$

$= -2-(-1)$

$= -2+1 = -1$

(c) $\lim_{x\to \frac{\pi}{2}^+} \sec x = \lim_{\substack{x\to\frac{\pi}{2} \\ x>\frac{\pi}{2}}} \left(\dfrac{1}{\cos x}\right)$

$= -\infty$



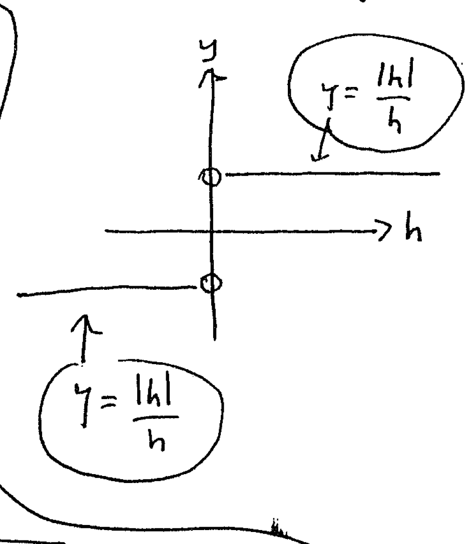
(d) If $f(x) = |x-4|$, then

$\lim_{h\to 0} \dfrac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \dfrac{|4+h-4| - |4-4|}{h} = \lim_{h\to 0} \dfrac{|h|}{h}$

$\therefore \lim_{h\to 0} \dfrac{|h|}{h}$ does not exist

Since $\lim_{h\to 0^+} \dfrac{|h|}{h} = 1 \neq -1 = \lim_{h\to 0^-} \dfrac{|h|}{h}$

$y = \dfrac{|h|}{h} = \begin{cases} \dfrac{h}{h} = 1, & \text{if } h>0 \\ \dfrac{-h}{h} = -1, & \text{if } h<0 \end{cases}$



(e) $\lim_{x\to 0} \dfrac{\tan 7x}{\sin 3x} = \lim_{x\to 0} \dfrac{\tan 7x \cdot \frac{7x}{7x}}{\sin 3x \cdot \frac{3x}{3x}}$

$= \lim_{x\to 0} \left(\dfrac{\frac{\tan 7x}{7x}}{\frac{\sin 3x}{3x}}\right)\left(\dfrac{7x}{3x}\right)$

$= \left(\dfrac{1}{1}\right)\cdot\dfrac{7}{3} = \dfrac{7}{3}$

Note:

$\lim_{x\to 0} \dfrac{\tan x}{x} = \lim_{x\to 0} \dfrac{\frac{\sin x}{\cos x}}{x}$

$= \lim_{x\to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}$

$= 1 \cdot \dfrac{1}{\cos(0)} = 1\cdot\dfrac{1}{1} = 1$

(f) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2} =$

$\lim_{x\to 0} \dfrac{1-\cos x}{x^2} \cdot \dfrac{1+\cos x}{1+\cos x} = \lim_{x\to 0} \dfrac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \dfrac{\sin^2 x}{x^2(1+\cos x)}$

$= \lim_{x\to 0} \left(\dfrac{\sin x}{x}\right)^2 \cdot \left(\dfrac{1}{1+\cos x}\right) = 1^2 \cdot \left(\dfrac{1}{1+1}\right) = 1\cdot\left(\dfrac{1}{2}\right) = \dfrac{1}{2}$

6. (Continued): Evaluate the following limits. **[No work, no credit.]**

(g) $\lim_{x\to-1}\frac{x^3-x^2+x+2}{x-1} = \frac{L_{x\to-1}(x^3-x^2+x+2)}{L_{x\to-1}(x-1)} = \frac{(-1)^3-(-1)^2+-1+2}{-1-1}$

$= \frac{-1-1-1+2}{-2} = \frac{-1}{-2} = \frac{1}{2}$

(h) $\lim_{x\to-\infty}\left(x+\sqrt{x^2-6x}\right) = L_{x\to-\infty}\ \frac{x+\sqrt{x^2-6x}}{1}\cdot\frac{x-\sqrt{x^2-6x}}{x-\sqrt{x^2-6x}}$

$= L_{x\to-\infty}\ \frac{x^2-(x^2-6x)}{x-\sqrt{x^2-6x}} = L_{x\to-\infty}\ \frac{6x}{x-\sqrt{x^2-6x}}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$

$= L_{x\to-\infty}\ \frac{6}{\frac{1}{x}\left[x-\sqrt{x^2\left(1-\frac{6x}{x^2}\right)}\right]} = L_{x\to-\infty}\ \frac{6}{\frac{1}{x}\left[x-\sqrt{x^2}\sqrt{1-\frac{6}{x}}\right]}$

$= L_{x\to-\infty}\ \frac{6}{\frac{x}{x}-\frac{|x|}{x}\sqrt{1-\frac{6}{x}}}$

$= L_{x\to-\infty}\ \frac{6}{1-\frac{(-x)}{x}\sqrt{1-\frac{6}{x}}}$ $\quad |x|=-x$ for $x<0$

$= L_{x\to-\infty}\ \frac{6}{1+\sqrt{1-\frac{6}{x}}}$

$= \frac{6}{1+\sqrt{1+0}} = \frac{6}{1+\sqrt{1}}$

$= \frac{6}{1+1} = \frac{6}{2} = 3$

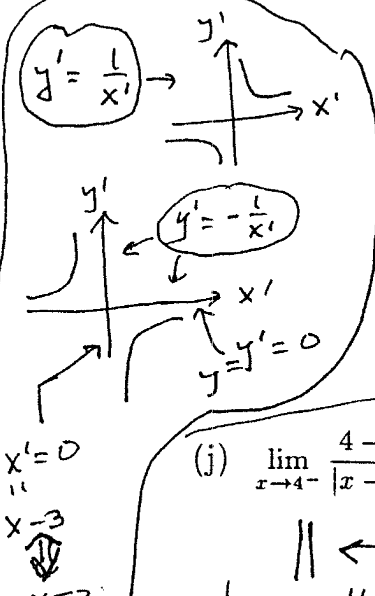
sign of $\frac{1}{3-x}$ = sign of $3-x$ for $x \neq 3$

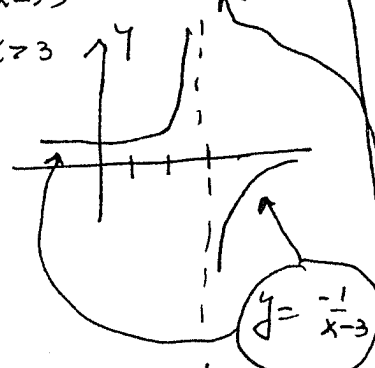
$-(x-3)$ ++++ 0 − − − −

$x-3$ − − − − 0 ++++

3

(i) $\lim_{x\to3^+}\frac{1}{3-x} = L_{\substack{x\to3\\x>3}}\ \frac{-1}{x-3} = -\infty$





$L_{x\to3^+}\ \frac{1}{3-x} = -\infty$

Note: $\frac{1}{3-x}<0$ for $x>3$ & as $x\to3$ with $x>3$, $3-x\to0$

(j) $\lim_{x\to4^-}\frac{4-x}{|x-4|} =$

$= L_{\substack{x\to4\\x<4}}\ \frac{4-x}{-(x-4)}$

$L_{\substack{x\to4\\x<4}}\ \frac{4-x}{-x+4} = L_{\substack{x\to4\\x<4}}\ \frac{4-x}{4-x} = L_{\substack{x\to4\\x<4}}\ 1 = 1$

$|x-4| = \begin{cases} x-4, & \text{if } x-4\geq0 \\ -(x-4) & \text{if } x-4\leq0\end{cases} = \begin{cases} x-4, & \text{if } x\geq4 \\ -(x-4), & \text{if } x\leq4\end{cases}$

(k) $\lim_{x\to3}\frac{\sqrt{x-2}-1}{x-3} = L_{x\to3}\ \frac{\sqrt{x-2}-1}{x-3}\cdot\frac{\sqrt{x-2}+1}{\sqrt{x-2}+1}$

$= L_{x\to3}\ \frac{\left(\sqrt{x-2}\right)^2-1^2}{x-3}\cdot\frac{1}{\sqrt{x-2}+1} = L_{x\to3}\ \frac{x-2-1}{(x-3)(\sqrt{x-2}+1)}$

$= L_{x\to3}\ \frac{x-3}{x-3}\cdot\frac{1}{\sqrt{x-2}+1} = \frac{1}{\sqrt{3-2}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2}$

EXTRA-CREDIT: [12 points: But, any score over 100 will be truncated to 100.]

1. Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{-6x}{\sqrt{9x^2+4}}.$$

[6 points]

SOLUTION:

$$\lim_{x\to\pm\infty} \frac{-6x}{\sqrt{9x^2+4}} = \lim_{x\to\pm\infty} \frac{-6x}{\sqrt{9x^2\left(1+\frac{4}{9x^2}\right)}}$$

$$= \lim_{x\to\pm\infty} \frac{-6x}{\sqrt{(3x)^2}\sqrt{1+\frac{4}{9x^2}}} = \lim_{x\to\pm\infty} \frac{-6x}{|3x|\sqrt{1+\frac{4}{9x^2}}} = \lim_{x\to\pm\infty} \frac{-6x}{|3||x|\cdot\sqrt{1+\frac{4}{9x^2}}}$$

$$= \lim_{x\to\pm\infty} \frac{-6x}{3\cdot|x|\cdot\sqrt{1+\frac{4}{9x^2}}}, \text{ so}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{-6x}{3\cdot x\cdot\sqrt{1+\frac{4}{9x^2}}} = \lim_{x\to\infty} \frac{-2}{\sqrt{1+\frac{4}{9x^2}}} = \frac{-2}{\sqrt{1+0}} = \frac{-2}{1} = -2$$

while

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-6x}{3(-x)\cdot\sqrt{1+\frac{4}{9x^2}}} = \lim_{x\to-\infty} \frac{2}{\sqrt{1+\frac{4}{9x^2}}} = \frac{2}{\sqrt{1+0}} = \frac{2}{1} = 2$$

$\therefore$ $\boxed{y=-2}$ and $\boxed{y=2}$ are the horizontal asymptotes of the graph $y = f(x)$.

2. Find the vertical and oblique (= slant) asymptotes of the graph of the function

$$f(x) = \frac{x^3}{x^2+x-6} = \frac{x^3}{(x+3)\cdot(x-2)}$$

(*) So $\boxed{y = x-1}$ is the slant asymptote of the graph $y=f(x)$

[6 points]

SOLUTION: Hence the vertical asymptotes of the graph $y = f(x)$ are $\boxed{x=-3}$ & $\boxed{x=2}$

To find the slant asymptote divide x^3 by x^2+x-6 to find:

$$\begin{array}{r|l} & x - 1 \\ \hline x^2+x-6 & x^3 + 0\cdot x^2 + 0\cdot x + 0 \\ & x^3 + x^2 - 6x \\ \hline & -x^2 + 6x \\ & -x^2 - x + 6 \\ \hline & 7x - 6 \end{array}$$

$$\Rightarrow \frac{x^3}{x^2+x-6} = \underbrace{x-1}_{g(x)} + \underbrace{\frac{7x-6}{x^2+x-6}}_{h(x)} =: g(x) + h(x),$$

Since $(x-1)(x^2+x-6) + 7x-6 = \cdots = x^3$

Check this out!

Now

$$\lim_{x\to\pm\infty} \frac{7x-6}{x^2+x-6} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\pm\infty} \frac{\frac{7}{x}-\frac{6}{x^2}}{1+\frac{1}{x}-\frac{1}{x^2}} = \frac{\pm 0 - 0}{1 \pm 0 - 0} = \frac{0}{1} = 0$$

Since $f(x) = g(x) + h(x)$ and $\lim_{x\to\pm\infty} h(x) = \lim_{x\to\pm\infty} \left(\frac{7x-6}{x^2+x-6}\right) = 0$, it follows that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} g(x) + \lim_{x\to\pm\infty} h(x) = \lim_{x\to\pm\infty} x-1+0 = \lim_{x\to\pm\infty} x-1 \quad (*)$$