Math 2108

Def. $f(x) := e^x := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$\Downarrow$

$$f'(x) = 0 + 1 + \frac{2x}{2\cdot 1} + \frac{3x^2}{3\cdot 2!} + \frac{4x^3}{4\cdot 3!} + \frac{5x^4}{5\cdot 4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$=: e^x =: f(x)$$

---

i.e., $\frac{de^x}{dx} = e^x$, so $\frac{de^u}{dx} = \frac{de^u}{du}\cdot\frac{du}{dx} = e^u\frac{du}{dx}$

i.e. $\left[e^{J(x)}\right]' = e^{J(x)}\cdot J'(x)$

$f(x) = e^{4x^2+3x+1} \Rightarrow f'(x) = e^{4x^2+3x+1}\cdot(8x+3)$

$f(x) = \cos(e^{2x}) \Rightarrow f'(x) = -\sin(e^{2x})\cdot e^{2x}\cdot 2$

$= -2\cdot e^{2x}\cdot\sin(e^{2x})$

$f(x) = \left[\cos(e^{2x})\right]^4$

$\Downarrow$

$f'(x) = 4\cdot\left[\cos(e^{2x})\right]^3\cdot\left[-\sin(e^{2x})\cdot e^{2x}\cdot 2\right]$

$f(x) = \sin x\cdot\cos x$

$\Downarrow$

$f'(x) = \sin x\cdot[-\sin x] + \cos x\cdot[\cos x]$

$= \cos^2 x - \sin^2 x = \frac{1}{2}\cdot 2\cdot\sin x\cdot\cos x = \frac{1}{2}\sin(2x)$

$\boxed{\text{OR}}$ $f(x) = \sin x\cdot\cos x = \frac{1}{2}\cdot 2\cdot\sin x\cdot\cos x = \frac{1}{2}\cdot\sin(2x)$

$\Downarrow$

$f'(x) = \frac{1}{2}\cdot\cos(2x)\cdot 2 = \cos(2x) = \cos^2(x) - \sin^2(x).$

<u>Note</u>: Recall that

$$\sin(A+B) = \sin(A)\cdot\cos(B) + \cos(A)\cdot\sin(B)$$

So, upon taking $A = B = x$, we have that

$$\sin(2x) = \sin(x+x) = \sin x \cdot \cos x + \cos x \cdot \sin x$$
$$= 2\sin x \cdot \cos x$$

Also,

$$\cos(A+B) = \cos(A)\cdot\cos(B) - \sin(A)\cdot\sin(B)$$

So, upon taking $A = B = x$, we have that

$$\cos(2x) = \cos(x+x) = \cos x \cdot \cos x - \sin x \cdot \sin x$$
$$= \cos^2 x - \sin^2 x$$

Further, since $\sin^2 x + \cos^2 x = 1$

we have that

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= 1 - \sin^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

and also that

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1 - \cos^2 x)$$
$$= 2\cos^2 x - 1$$

which leads to the <u>$\frac{1}{2}$-angle formulas</u>

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \& \quad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

i.e.,

$$\cos^2\left(\frac{x}{2}\right) = \frac{1 + \cos(x)}{2} \quad \& \quad \sin^2\left(\frac{x}{2}\right) = \frac{1 - \cos(x)}{2}.$$

Note: Be sure not to confuse

$$f(x) = \sin x \cdot \cos x = \cos x \cdot \sin x$$

with either

$$f(x) = \sin[\cos x] \quad \underline{\text{or}} \quad f(x) = \cos[\sin x]$$

Indeed

$$f(x) = \sin x \cdot \cos x$$

$$\Downarrow$$

$$\begin{aligned} f'(x) &= \sin x \cdot (\cos x)' + \cos x \cdot (\sin x)' \\ &= \sin x \cdot (-\sin x) + \cos x \cdot \cos x \\ &= \cos^2 x - \sin^2 x, \end{aligned}$$

as we saw above, whereas

$$f(x) = \sin[\cos x]$$

$$\Downarrow$$

$$\begin{aligned} f'(x) &= \cos[\cos x] \cdot (\cos x)' \\ &= \cos[\cos x] \cdot (-\sin x) \\ &= -\sin x \cdot \cos[\cos x] \end{aligned}$$

while $f(x) = \cos[\sin x]$

$$\Downarrow$$

$$\begin{aligned} f'(x) &= -\sin[\sin x] \cdot (\sin x)' \\ &= -\sin[\sin x] \cdot \cos x \\ &= -\cos x \cdot \sin[\sin x]. \end{aligned}$$

Indeed, one has

$$(f \cdot g)'(x) = \left(f(x) \cdot g(x)\right)'$$
$$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

versus

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

Recall: $(f \cdot g)(x) := f(x) \cdot g(x)$
= the pointwise product of $f(x)$ & $g(x)$

whereas

$$f \circ g\,(x) := f[g(x)]$$

i.e., $f \circ g$ = the composite of the <u>inner function</u> g followed by the <u>outer function</u> f

i.e., $f \cdot g$ = the pointwise product of f & g

while $f \circ g$ = the composition product of f ~~&~~ g

Junk of x

$f(x) = \sin[J(x)]$

$\Downarrow$

$f'(x) = \cos[J(x)] \cdot J'(x)$

$f(x) = \cos[J(x)]$

$\Downarrow$

$f'(x) = -\sin[J(x)] \cdot J'(x)$

$\boxed{\text{OR}}$

$y = f(x) = \sin(u)$

$\Downarrow$

$\frac{dy}{dx} = f'(x) = \cos(u) \cdot \frac{du}{dx}$

$y = f(x) = \cos(u)$

$\Downarrow$

$\frac{dy}{dx} = f'(x) = -\sin(u) \frac{du}{dx}$

$\boxed{\text{OR}}$

$y = \sin(u)$

$\Downarrow$

$\frac{dy}{dx} = \frac{d \sin(u)}{dx}$

$= \frac{d \sin(u)}{du} \cdot \frac{du}{dx}$

$= \cos(u) \cdot \frac{du}{dx}$

$y = \cos(u)$

$\Downarrow$

$\frac{dy}{dx} = \frac{d \cos(u)}{dx}$

$= \frac{d \cos(u)}{du} \cdot \frac{du}{dx}$

$= -\sin(u) \frac{du}{dx}$

$$y = f(x) = \tan[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = \sec^2[J(x)] \cdot J'(x)$$

$\boxed{\text{OR}}$ $\quad y = \tan(u)$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d\tan(u)}{dx} = \frac{d\tan(u)}{du}\frac{du}{dx}$$

$$= \sec^2(u) \cdot \frac{du}{dx}$$

$$y = f(x) = \cot[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = -\csc^2[J(x)] \cdot J'(x)$$

$\boxed{\text{OR}}$ $\quad y = \cot(u)$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d\cot(u)}{dx} = \frac{d\cot(u)}{du} \cdot \frac{du}{dx}$$

$$= -\csc^2(u) \cdot \frac{du}{dx}$$

$$y = f(x) = \sec[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = \sec[J(x)] \cdot \tan[J(x)] \cdot J'(x)$$

$\boxed{\text{OR}}$

$$y = \sec(u)$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d\,\sec(u)}{dx} = \frac{d\,\sec(u)}{du} \cdot \frac{du}{dx}$$

$$= \sec(u) \cdot \tan(u) \cdot \frac{du}{dx}$$

$$y = f(x) = \csc[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = -\csc[J(x)] \cdot \cot[J(x)] \cdot J'(x)$$

$\boxed{\text{OR}}$

$$y = \csc(u)$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d\,\csc(u)}{dx} = \frac{d\,\csc(u)}{du} \cdot \frac{du}{dx}$$

$$= -\csc(u) \cdot \cot(u) \cdot \frac{du}{dx}$$

$$y = f(x) = e^{J(x)}$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = e^{J(x)} \cdot J'(x)$$

$\boxed{\text{OR}}$

$$y = e^{u}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d e^{u}}{dx} = \frac{d e^{u}}{du} \cdot \frac{du}{dx}$$

$$= e^{u} \cdot \frac{du}{dx}$$

$$y = \ln[J(x)] = \log_e[J(x)] \quad , \text{ for } J(x) > 0$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = \frac{1}{J(x)} \cdot J'(x) = \frac{J'(x)}{J(x)}$$

$\boxed{\text{OR}}$ $y = \ln(u)$ for $u > 0$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d \ln(u)}{dx} = \frac{d \ln(u)}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot \frac{du}{dx}$$

$y = f(x) = [J(x)]^c$, where c = a constant

$\Downarrow$

$\frac{dy}{dx} = f'(x) = c \cdot [J(x)]^{c-1} \cdot J'(x)$

$\boxed{\text{OR}}$

$y = u^c$

a constant

a variable = a function of x

$\Downarrow$

$\frac{dy}{dx} = \frac{du^c}{dx} = \frac{du^c}{du} \cdot \frac{du}{dx}$

$= c \cdot u^{c-1} \cdot \frac{du}{dx}$

$$f(x) = \sin\left[\cos(x^2)\right] = \sin\left[J(x)\right]$$

$$\Downarrow$$

$$f'(x) = \cos\left[J(x)\right] \cdot J'(x)$$

or

$$f(x) = \sin(u)$$

$$\Downarrow$$

$$f'(x) = \cos(u) \cdot \frac{du}{dx}$$

So:

$$f(x) = \sin\left[\cos(x^2)\right]$$

$$\Downarrow$$

$$f'(x) = \cos\left[\cos(x^2)\right] \cdot \left[-\sin(x^2) \cdot (2x)\right]$$

OR

$$y = \sin\left[\cos(x^2)\right]$$

$$\frac{dy}{dx} = \frac{d \sin\left[\cos(x^2)\right]}{dx}$$

$$= \frac{d \sin\left[\cos(x^2)\right]}{d \cos(x^2)} \cdot \frac{d \cos(x^2)}{dx}$$

$$= \cos\left[\cos(x^2)\right] \cdot \frac{d \cos(x^2)}{dx^2} \cdot \frac{dx^2}{dx}$$

$$= \cos\left[\cos(x^2)\right] \cdot \left[-\sin(x^2) \cdot 2x\right]$$

11

$$f(x) = \left(\sin\left[\cos(x^2)\right]\right)^3 = \left(J(x)\right)^3$$

$$\Downarrow$$

$$f'(x) = 3 \cdot \left(J(x)\right)^2 \cdot J'(x)$$

$$= 3 \cdot \left(\sin\left[\cos(x^2)\right]\right)^2 \cdot \left[\cos\left[\cos(x^2)\right] \cdot \left(-\sin(x^2) \cdot 2x\right)\right]$$

$\boxed{\text{OR}}$

$y = f(x) = u^3$ where $u = \sin\left[\cos(x^2)\right]$

$= \sin(v)$ where

$v = \cos(x^2)$

$= \cos(w)$

where

$w = x^2$

$$\Downarrow$$

$$f'(x) = \frac{dy}{dx} = \frac{d\,u^3}{dx} = \frac{du^3}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \frac{du}{dx}$$

$$= 3u^2 \cdot \frac{d\sin(v)}{dx} = 3u^2 \cdot \frac{d\sin(v)}{dv} \cdot \frac{dv}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot \frac{dv}{dx} = 3u^2 \cdot \cos(v) \cdot \frac{d\cos(w)}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot \frac{d\cos(w)}{dw} \cdot \frac{dw}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot \left(-\sin(w)\right) \cdot \frac{dw}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot \left(-\sin(w)\right) \cdot \frac{dx^2}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot \left(-\sin(w)\right) \cdot 2x$$

$$= 3\left(\sin\left[\cos(x^2)\right]\right)^2 \cdot \cos\left[\cos(x^2)\right] \cdot \left(-\sin(x^2)\right) \cdot 2x$$

$$= -6x \cdot \left(\sin\left[\cos(x^2)\right]\right)^2 \cdot \cos\left[\cos(x^2)\right] \cdot \sin(x^2)$$

$$f(x) = x^7 + x^5 \Rightarrow f'(x) = 7x^6 + 5x^4$$

$$f(x) = -6x^3 + 12x^2 - 4x + 7$$
$$\Downarrow$$
$$f'(x) = -18x^2 + 24x - 4$$

$$f(x) = \frac{1}{10}\cdot x^{10} + \frac{1}{9}\cdot x^9 + \frac{1}{8}\cdot x^8$$
$$\Downarrow$$
$$f'(x) = x^9 + x^8 + x^7$$

$$f(x) = (3x+5)^2 = 9x^2 + 30x + 25$$
$$\Downarrow$$
$$f'(x) = 18x + 30$$

$\boxed{\text{OR}}$ $f(x) = (3x+5)^2 \Rightarrow f'(x) = 2\cdot(3x+5)'\cdot 3$
$$= 6\cdot(3x+5) = 18x + 30$$

$$f(x) = (-2x^2+1)^3 = (-2x^2+1)\cdot(-2x^2+1)^2$$
$$= (-2x^2+1)\cdot(4x^4 - 4x^2 + 1)$$
$$= -8x^6 + 8x^4 - 2x^2 + 4x^4 - 4x^2 + 1$$
$$= -8x^6 + 12x^4 - 6x^2 + 1$$
$$\Downarrow$$
$$f'(x) = -48x^5 + 48x^3 - 12x$$

$\boxed{\text{OR}}$ $f(x) = (-2x^2+1)^3 = (a+b)^3$ where $a = -2x^2$ & $b = 1$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$
$$= a^3 + 3a^2 + 3a + 1$$
$$= (-2x^2)^3 + 3\cdot(-2x^2)^2 + 3\cdot(-2x^2) + 1$$
$$= -8x^6 + 3\cdot 4x^4 - 6x^2 = -8x^6 + 12x^4 - 6x^2 + 1$$
$$\Downarrow$$
$$f'(x) = -48x^5 + 48x^3 - 12x$$

$\boxed{\text{OR}}$ $f(x) = (-2x^2+1)^3 \Rightarrow f'(x) = 3\cdot(-2x^2+1)^2\cdot(-4x)$
$$= -12x\cdot(4x^4 - 4x^2 + 1)$$
$$= -48x^5 + 48x^3 - 12x$$