Math 2107

Theorem ①: If $f:\mathbb{R}\to\mathbb{R}$ is a function & $a\in\mathbb{R}$, then f is continuous at a whenever $f'(a)$ exists.

Proof:

$$\lim_{h\to 0} f(a+h) = \lim_{h\to 0}\left\{ h\cdot\left[\frac{f(a+h)-f(a)}{h}\right] + f(a)\right\}$$

$$= \lim_{h\to 0} h \cdot \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \lim_{h\to 0} f(a)$$

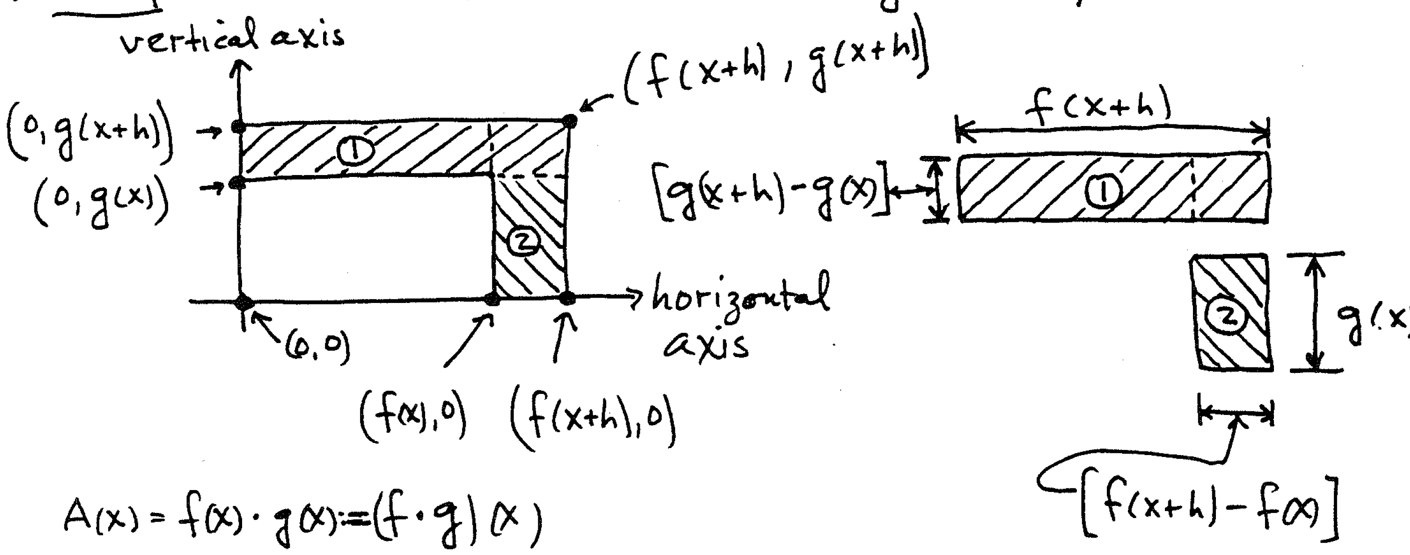
$$= 0 \cdot f'(a) + f(a)$$

$$= f(a)$$

Theorem ②: If $f'(x)$ exists & $g'(x)$ exists, then $(f\cdot g)'(x)$ exists
&
$$(f\cdot g)'(x) = f(x)\cdot g'(x) + g(x)\cdot f'(x).$$

[If F = first & S = second, $(F\cdot S)' = F\cdot S' + S\cdot F'$]

1st **Proof**: Let $A(x) := \text{Area}(x) := f(x)\cdot g(x) = \text{Length}\cdot\text{Width}$



$$A(x) = f(x)\cdot g(x) := (f\cdot g)(x)$$

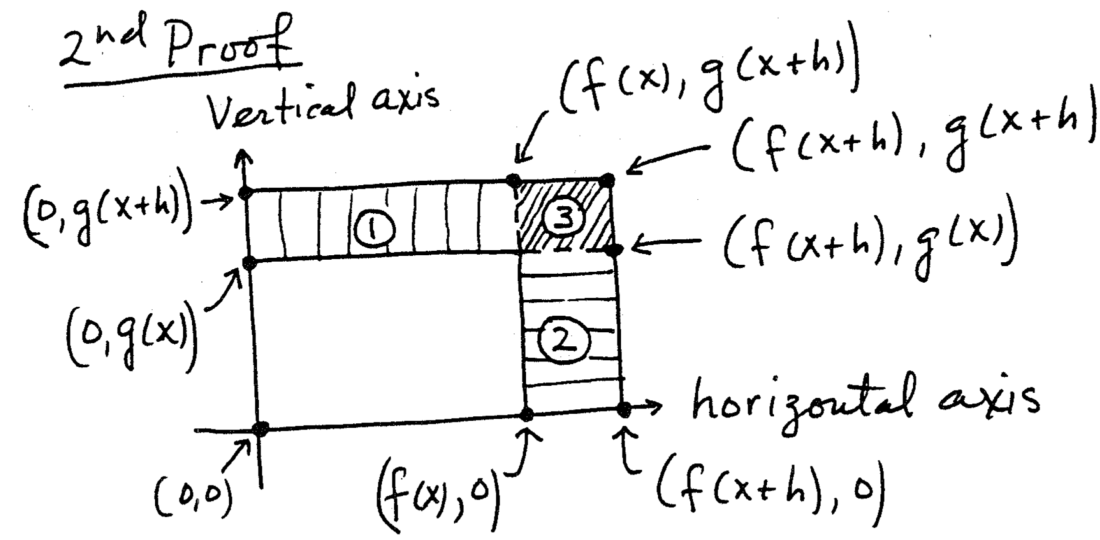
$$\frac{A(x+h)-A(x)}{h} = \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

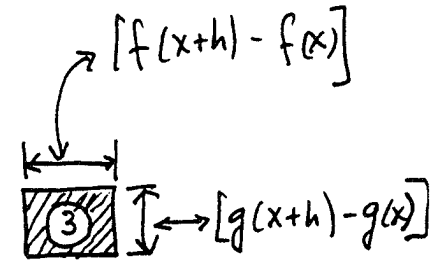
$$= \frac{f(x+h)\cdot[g(x+h)-g(x)] + g(x)\cdot[f(x+h)-f(x)]}{h}$$

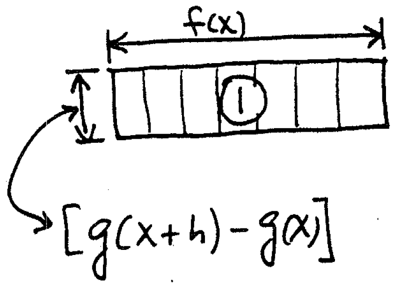
$$(f\cdot g)'(x) = A'(x) = \text{Limit}_{h\to 0}\left\{ f(x+h)\cdot\left[\frac{g(x+h)-g(x)}{h}\right] + g(x)\cdot\left[\frac{f(x+h)-f(x)}{h}\right]\right\}$$

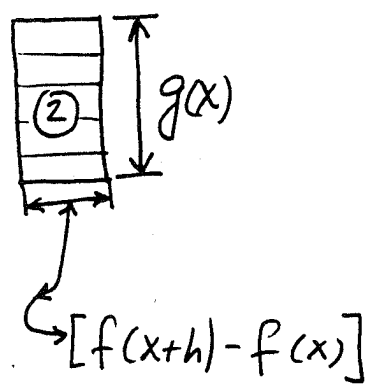
by Theorem ① $= f'(x)\cdot g'(x) + g(x)\cdot f'(x).$

2nd Proof









Hence, with $A(x) := (f \cdot g)(x) := f(x) \cdot g(x)$, we have

$$A(x+h) - A(x) = f(x) \cdot [g(x+h) - g(x)] + g(x) \cdot [f(x+h) - f(x)] + [f(x+h) - f(x)] \cdot [g(x+h) - g(x)]$$

so

$$\frac{A(x+h) - A(x)}{h} = f(x) \cdot \left[\frac{g(x+h) - g(x)}{h}\right] + g(x) \cdot \left[\frac{f(x+h) - f(x)}{h}\right] + \left[\frac{f(x+h) - f(x)}{h}\right] \cdot \left[\frac{g(x+h) - g(x)}{h}\right] \cdot h$$

so (upon taking limits of both sides as $h \to 0$) we find that:

$$(f \cdot g)'(x) = A'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x) + f'(x) \cdot g'(x) \cdot 0$$
$$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

<u>Note</u>: This theorem does <u>not</u> require Theorem 1 !!!!!

Theorem 3: If $g'(x)$ exists & $g(x) \neq 0$, then $\left(\frac{1}{g}\right)'(x)$ exists & $\left(\frac{1}{g}\right)'(x) = \frac{-1}{[g(x)]^2} \cdot g'(x)$.

1st Proof:

$$\frac{\frac{1}{g}(x+h) - \frac{1}{g}(x)}{h} := \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \frac{g(x) - g(x+h)}{g(x+h) \cdot g(x) \cdot h}$$

$$= \frac{-1}{g(x+h) \cdot g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$\therefore \left(\frac{1}{g}\right)'(x) = \lim_{h \to 0} \frac{\frac{1}{g}(x+h) - \frac{1}{g}(x)}{h} = \frac{-1}{g(x) \cdot g(x)} \cdot g'(x)$$

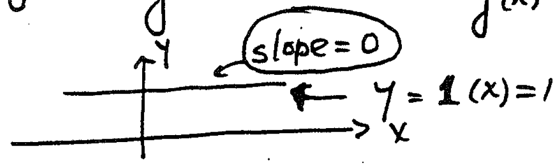
By Theorem 1

$$= \frac{-1}{[g(x)]^2} \cdot g'(x)$$

2nd Proof: 1st note that $g \cdot \frac{1}{g} = \mathbf{1}$ = the constant function $\mathbf{1} : \mathbb{R} \to \mathbb{R}$ (with $\mathbf{1}(x) = 1$ for all x)

since $\left(g \cdot \frac{1}{g}\right)(x) := g(x) \cdot \frac{1}{g}(x) := g(x) \cdot \frac{1}{g(x)} = 1$ for all x.

Now $\mathbf{1}'(x) = 0$:



Hence $0 = \mathbf{1}'(x) = \left(g \cdot \frac{1}{g}\right)'(x) = g(x) \cdot \left(\frac{1}{g}\right)'(x) + \frac{1}{g}(x) \cdot g'(x)$

$$:= g(x) \cdot \left(\frac{1}{g}\right)'(x) + \frac{1}{g(x)} \cdot g'(x)$$

so

$$g(x) \cdot \left(\frac{1}{g}\right)'(x) = -\frac{1}{g(x)} \cdot g'(x)$$

so

$$\left(\frac{1}{g}\right)'(x) = -\frac{1}{[g(x)]^2} \cdot g'(x)$$

<u>Note</u>: This proof, combined with the 2nd proof of Theorem 2, does <u>not</u> require Theorem 1.

Theorem (4): If $f'(x)$ exists & $g'(x)$ exists & $g(x) \neq 0$ then $\left(\frac{f}{g}\right)'(x)$ exists & $\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

i.e.,
$$\left(\frac{T}{B}\right)' = \frac{B \cdot T' - T \cdot B'}{B^2}$$

[The derivative of Top over Bottom equals the Bottom times the derivative of the Top MINUS the Top times the derivative of the Bottom, all over the Bottom squared.]

ADVICE: MEMORIZE THIS LIKE A POEM

1st Proof:
$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f \cdot \left(\frac{1}{g}\right)' + \frac{1}{g} \cdot f'$$
$$= f \cdot \left[\frac{-1}{g^2} \cdot g'\right] + \frac{1}{g} \cdot f'$$
$$= f \left[-\frac{1}{g^2} \cdot g'\right] + \frac{g}{g} \cdot \frac{1}{g} \cdot f'$$
$$= \frac{g \cdot f' - f \cdot g'}{g^2}$$

2nd Proof: Set $Q = \frac{T}{B}$. Then
(not requiring Theorem 1)

$T = Q \cdot B$, so
$$T' = (Q \cdot B)' = Q \cdot B' + B \cdot Q'$$

Now "Solve" for Q': $\quad BQ' = T' - Q \cdot B' = T' - \frac{T}{B} \cdot B$

so
$$\left(\frac{T}{B}\right)' = Q' = \frac{1}{B} \cdot \left[T' - \frac{T}{B} \cdot B'\right] = \frac{1}{B} \cdot \left[\frac{B}{B} \cdot T' - \frac{T}{B} \cdot B'\right]$$
$$= \frac{B \cdot T' - T \cdot B'}{B^2}$$

Power Rules: Define

$$f^2 = f \cdot f$$
$$f^3 = f \cdot f^2$$
$$f^4 = f \cdot f^3$$
etc.
$$\vdots$$
$$f^{n+1} = f \cdot f^n \quad \text{for } n \geq 1$$

Then $$(f^2)' = (f \cdot f)' = f \cdot f' + f \cdot f' = 2 f \cdot f'$$

$$(f^3)' = (f \cdot f^2)' = f \cdot (f^2)' + f^2 \cdot f'$$
$$= f \cdot 2f \cdot f' + f^2 \cdot f'$$
$$= 2 f^2 \cdot f' + f^2 \cdot f'$$
$$= (2+1) \cdot f^2 \cdot f'$$
$$= 3 f^2 \cdot f'$$

$$\vdots$$

Suppose it has been shown that
$$(f^n)' = n \cdot f^{n-1} \cdot f'$$

Then
$$(f^{n+1})' = (f \cdot f^n)' = f \cdot (f^n)' + f^n \cdot f'$$
$$= f \cdot n f^{n-1} \cdot f' + f^n \cdot f'$$
$$= n f^n \cdot f' + f^n \cdot f'$$
$$= (n+1) f^n \cdot f'$$

What about roots?

Suppose

$$f(x) = \sqrt{g(x)} \in \mathbb{R} = \text{the set of all real numbers}$$

Then $g(x) \geq 0$. Of course if $g(x) = 0$ then $f(x) = \sqrt{0} = 0$ & all is trivial; so we may as well suppose that

$$f(x) = \sqrt{g(x)} \quad \text{with } g(x) > 0$$

In particular, then, $g(x) \neq 0$.
Suppose further that $g'(x)$ exists.

Claim: $f'(x)$ exists &

$$f'(x) = \frac{1}{2\sqrt{g(x)}} \cdot g'(x)$$

i.e., if $f(x) = [g(x)]^{1/2}$, then

$$f'(x) = \frac{1}{2}[g(x)]^{-1/2} \cdot g'(x).$$

Proof: $f(x) = \sqrt{g(x)} \Rightarrow f(x) \cdot f(x) = \sqrt{g(x)} \cdot \sqrt{g(x)} = g(x)$.

If we knew $f'(x)$ existed, we would find that

$$g'(x) = f(x) \cdot f'(x) + f(x) \cdot f'(x) = 2f(x) \cdot f'(x)$$

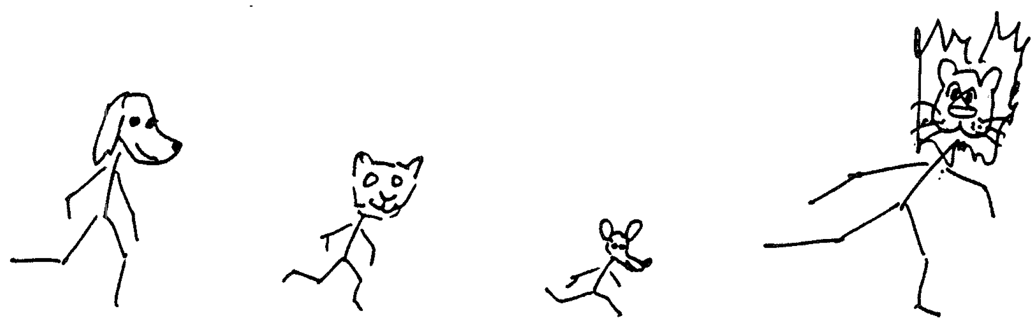
so $$f'(x) = \frac{1}{2f(x)} \cdot g'(x) = \frac{1}{2\sqrt{g(x)}} \cdot g'(x).$$

But, for geometric reasons the inverse of a differentiable function is differentiable & $f = k^{-1}$ where $k(x) = x^2$, so f' exists!

Math 2107

The Chain Rule: Frequently one encounters a situation in which we have a chain of events each of which depends on the preceding event in such a way that the rate of the last event with respect to (abbreviated: w.r.t.) the 1st event is the product of the relative rates of the intervening events.

To be specific, suppose a dog chases a cat who in turn chases a mouse who in turn chases a lion!



Suppose that

1) The cat takes 4 strides for each stride of the dog. Then, the rate of the cat w.r.t. the dog = 4:

$$\frac{d\ \text{Cat}}{d\ \text{Dog}} = 4.$$

2) The mouse takes 7 strides for each stride of the cat. Then, the rate of the mouse w.r.t. the cat = 7:

$$\frac{d\ \text{Mouse}}{d\ \text{Cat}} = 7,$$

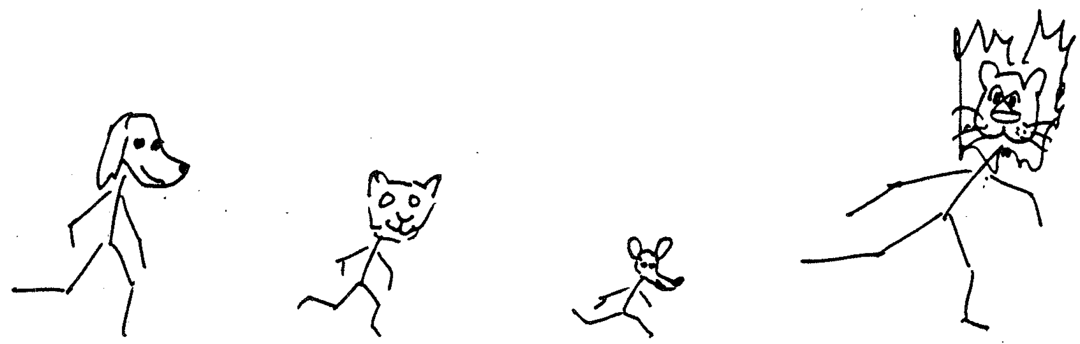
3) The lion takes 3 strides for each stride of the mouse. Then, the rate of the lion w.r.t. the mouse = 3:

$$\frac{d\ \text{Lion}}{d\ \text{Mouse}} = 3.$$

Question: How many strides does the Lion take for each stride of the dog? i.e.,
What is the rate of the lion w.r.t. the dog?
i.e., What is

$$\frac{d\,\text{Lion}}{d\,\text{Dog}} \quad ?$$

Answer:



The rate of the lion w.r.t. the dog clearly is the product of the 3 intervening rates:

1) the rate of the lion w.r.t. the mouse

times

2) the rate of the mouse w.r.t. the cat

times

3) the rate of the cat w.r.t. the dog:

$$\frac{d\,\text{Lion}}{d\,\text{Dog}} = \frac{d\,\text{Lion}}{d\,\text{Mouse}} \cdot \frac{d\,\text{Mouse}}{d\,\text{Cat}} \cdot \frac{d\,\text{Cat}}{d\,\text{Dog}}$$

$$= 3 \cdot 7 \cdot 4$$

$$= 84$$

§6. THE CHAIN RULE

We know how to build up new functions from old ones by means of sums, products, and quotients. There is one other important way of building up new functions. We shall first give examples of this new way.

Consider the function $(x + 2)^{10}$. We can say that this function is made up of the 10th power function, and the function $x + 2$. Namely, given a number x, we first add 2 to it, and then take the 10th power. Let

$$g(x) = x + 2$$

and let f be the 10th power function. Then we can take the value of f at $x + 2$, namely

$$f(x + 2) = (x + 2)^{10}$$

and we can also write it as

$$f(x + 2) = f(g(x)).$$

Another example: Consider the function $(3x^4 - 1)^{1/2}$. If we let $g(x) = 3x^4 - 1$ and f be the square root function, then

$$f(g(x)) = \sqrt{3x^4 - 1} = (3x^4 - 1)^{1/2}.$$

In order not to get confused by the letter x, which cannot serve us any more in all contexts, we use another letter to denote the values of g. Thus we may write $f(u) = u^{1/2}$.

Similarly, let $f(u)$ be the function $u + 5$ and $g(x) = 2x$. Then

$$f(g(x)) = f(2x) = 2x + 5.$$

One more example of the same type: Let

$$f(u) = \frac{1}{u + 2}$$

and

$$g(x) = x^{10}.$$

Then

$$f(g(x)) = \frac{1}{x^{10} + 2}.$$

In order to give you sufficient practice with many types of functions, we now mention several of them whose definitions will be given later. These will be sin and cos (which we read sine and cosine), log (which we read logarithm or simply log), and the exponential function exp. We shall select a special number e (whose value is approximately 2.718...), such that the function exp is given by

$$\exp(x) = e^x.$$

We now see how we make new functions out of these:

Let $f(u) = \sin u$ and $g(x) = x^2$. Then

$$f(g(x)) = \sin(x^2).$$

Let $f(u) = e^u$ and $g(x) = \cos x$. Then

$$f(g(x)) = e^{\cos x}.$$

Let $f(v) = \log v$ and $g(t) = t^3 - 1$. Then

$$f(g(t)) = \log(t^3 - 1).$$

Let $g(w) = w^{10}$ and $f(z) = \log z + \sin z$. Then

$$g(f(z)) = (\log z + \sin z)^{10}.$$

Whenever we have two functions f and g such that f is defined for all numbers which are values of g, then we can build a new function denoted by $f \circ g$ whose value at a number x is

$$(f \circ g)(x) = f(g(x)).$$

The rule defining this new function is: Take the number x, find the number $g(x)$, and then take the value of f at $g(x)$. This is the value of $f \circ g$ at x. The function $f \circ g$ is called the **composite function** of f and g. We say that g is the **inner** function and that f is the **outer** function. For example, in the function log sin x, we have

$$f \circ g = \log \circ \sin.$$

The outer function is the log, and the inner function is the sine.

It is important to keep in mind that we can compose two functions only when the outer function is defined at all values of the inner function. For instance, let $f(u) = u^{1/2}$ and $g(x) = -x^2$. Then we cannot form the composite function $f \circ g$ because f is defined only for positive numbers (or 0) and the values of g are all negative, or 0. Thus $(-x^2)^{1/2}$ does not make sense.

However, for the moment you are asked to learn the mechanism of composite functions just the way you learned the multiplication table, in order to acquire efficient conditioned reflexes when you meet composite functions. Hence for the drills given by the exercises at the end of the section, you should forget for a while the meaning of the symbols and operate with them formally, just to learn the formal rules properly.

Even though we have not defined the exponential function e^x nor have we dealt formally with sin x or other of the functions just mentioned, nevertheless, we don't need to know their definitions in order to manipulate them. If we limited ourselves only to those functions which we have explicitly dealt with

so far, we would be restricted in seeing how composite functions work, and how the chain rule works below.

We come to the problem of taking the derivative of a composite function.

We start with an example. Suppose we want to find the derivative of the function $(x + 1)^{10}$. The Newton quotient would be a very long expression, which it would be essentially hopeless to disentangle by brute force, the way we have up to now. It is therefore a pleasant surprise that there will be an easy way of finding the derivative. We tell you the answer right away: The derivative of this function is $10(x + 1)^9$. This looks very much related to the derivative of powers.

Before stating and proving the general theorem, we give you other examples.

Example.

$$\frac{d}{dx}(x^2 + 2x)^{3/2} = \frac{3}{2}(x^2 + 2x)^{1/2}(2x + 2)$$

Observe carefully the extra term $2x + 2$, which is the derivative of the expression $x^2 + 2x$. We may describe the answer in the following terms. We let $u = x^2 + x$ so that $du/dx = 2x + 2$. Let $f(u) = u^{3/2}$. Then we have

$$\frac{d(f(u(x)))}{dx} = \frac{df}{du}\frac{du}{dx}.$$

Example.

$$\frac{d}{dx}(x^2 + x)^{10} = 10(x^2 + x)^9(2x + 1).$$

Observe again the presence of the term $2x + 1$, which is the derivative of $x^2 + x$. Again, if we let $u = x^2 + x$ and $f(u) = u^{10}$, then

$$\frac{df(u(x))}{dx} = \frac{df}{du}\frac{du}{dx}, \quad \text{where} \quad \frac{df}{du} = 10u^9 \quad \text{and} \quad \frac{du}{dx} = 2x + 1.$$

Can you guess the general rule from the preceding assertions? The general rule was also discovered by trial and error, but we profit from three centuries of experience, and thus we are able to state it and prove it very simply, as follows.

Chain Rule. *Let f and g be two functions having derivatives, and such that f is defined at all numbers which are values of g. Then the composite function $f \circ g$ has a derivative, given by the formula*

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

This can be expressed in words by saying that we take the *derivative of the outer function times the derivative of the inner function* (*or the derivative of what's inside*).

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The preceding assertion is known as the **chain rule.** If we put $u = g(x)$, then we may express the chain rule in the form

$$\boxed{\frac{df(u(x))}{dx} = \frac{df}{du}\frac{du}{dx},}$$

or also

$$\frac{d(f \circ g)}{dx} = \frac{df}{du}\frac{du}{dx}.$$

Thus the derivative behaves *as if* we could cancel the du. As long as we have proved this result, there is nothing wrong with working like a machine in computing derivatives of composite functions, and we shall give you several examples before the exercises.

Example. Let $f(u) = u^{10}$ and $u = g(x) = x^2 + 1$. Then $f'(u) = 10u^9 = df/du$ and $g'(x) = 2x = dg/dx$. Thus

$$\frac{d(f \circ g)}{dx} = \frac{df}{du}\frac{du}{dx} = 10u^9 \cdot 2x = 10(x^2 + 1)^9 2x.$$

Example. Let $f(u) = 2u^{1/2}$ and $g(x) = 5x + 1$. Then $f'(u) = u^{-1/2} = df/du$ and $g'(x) = 5 = dg/dx$. Thus

$$\frac{d(f \circ g)}{dx} = \frac{df}{du}\frac{du}{dx} = (5x + 1)^{-1/2} \cdot 5.$$

(Pay attention to the constant 5, which is the derivative of $5x + 1$. You are very likely to forget it.)

In order to give you more extensive drilling than would be afforded by the functions we have considered, like powers, we summarize the derivatives of the elementary functions which are to be considered later.

$$\frac{d(\sin x)}{dx} = \cos x. \qquad \frac{d(\cos x)}{dx} = -\sin x.$$

$$\frac{d(e^x)}{dx} = e^x \text{ (yes, } e^x\text{, the same as the function!).}$$

$$\frac{d(\log x)}{dx} = \frac{1}{x}.$$

Example.

$$\frac{d}{dx}(\sin x)^7 = 7(\sin x)^6 \cos x.$$

In this example, $f(u) = u^7$ and $df/du = 7u^6$. Also $u = \sin x$, and $du/dx = \cos x$.

Example.

$$\frac{d}{dx}(\sin 3x)^7 = 7(\sin 3x)^6 \cos 3x \cdot 3.$$

The last factor of 3 occurring on the right-hand side is the derivative $d(3x)/dx$.

Example. Let n be any integer. For any differentiable function $f(x)$,

$$\boxed{\frac{d}{dx}f(x)^n = nf(x)^{n-1}\frac{df}{dx}.}$$

Example.

$$\frac{de^{\sin x}}{dx} = \frac{df}{du}\frac{du}{dx} = e^{\sin x}(\cos x).$$

In this example, $f(u) = e^u$ and $df/du = e^u$.

Example.

$$\frac{d\cos(2x^2)}{dx} = \frac{df}{du}\frac{du}{dx} = -\sin(2x^2) \cdot 4x.$$

In this example, $f(u) = \cos u$ and $df/du = -\sin u$. Also $u = 2x^2$ so that $du/dx = 4x$.

Example.

$$\frac{d\cos 4x}{dx} = -\sin(4x) \cdot 4.$$

In this example, $f(u) = \cos u$ and $u = 4x$ so $du/dx = 4$.

We emphasize what we have already stated. *If we limited ourselves just to polynomials or quotients of polynomials, we would not have enough examples to drill the mechanism of the chain rule. There is nothing wrong in using the properties of functions which have not yet been formally defined in the course.* We could in fact create totally imaginary functions to achieve the same end.

Example. Suppose there is a function called schmoo (x), whose derivative is given by

$$\frac{d \text{ schmoo}(x)}{dx} = \frac{1}{x + \sin x}.$$

Then

$$\frac{d}{dx}\text{schmoo}(x^3 + 4x) = \frac{d \text{ schmoo}(u)}{du}\frac{du}{dx}$$

$$= \frac{1}{(x^3 + 4x) + \sin(x^3 + 4x)}(3x^2 + 4).$$

Example. Suppose there is a function cow (x) such that cow$'(x) =$ schmoo (x). Then

$$\frac{d \text{ cow}(x^2)}{dx} = \text{schmoo}(x^2) \cdot 2x.$$

Proof of the Chain Rule. We must consider the Newton quotient of the composite function $f \circ g$. By definition, it is

$$\frac{f[g(x + h)] - f[g(x)]}{h}.$$

Put $u = g(x)$, and let

$$k = g(x + h) - g(x).$$

Then k depends on h, and tends to 0 as h approaches 0. Our Newton quotient is equal to

$$\frac{f(u + k) - f(u)}{h}.$$

For the present argument *suppose that k is unequal to* 0 for all small values of h. Then we can multiply and divide this quotient by k, and obtain

$$\frac{f(u + k) - f(u)}{k}\frac{k}{h} = \frac{f(u + k) - f(u)}{k}\frac{g(x + h) - g(x)}{h}.$$

If we let h approach 0 and use the rule for the limit of a product, we see that our Newton quotient approaches

$$f'(u)g'(x),$$

and this would prove our chain rule, under the assumption that k is not 0.

It does not happen very often that $k = 0$ for arbitrarily small values of h, but when it does happen, the preceding argument must be refined. For those of you who are interested, we shall show you how the argument can be slightly modified so as to be valid in all cases. ***The uninterested reader can just skip it.***

We distinguish two kinds of numbers h. The first kind, those for which $g(x + h) - g(x) \neq 0$, and the second kind, those for which

$$g(x + h) - g(x) = 0.$$

Let H_1 be the set of h of the first kind, and H_2 the set of h of the second kind. We assume that we have

$$g(x + h) - g(x) = 0$$

for arbitrarily small values of h. Then the Newton quotient

$$\frac{g(x + h) - g(x)}{h}$$

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is 0 for such values, that is for h in H_2, and consequently

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = 0.$$

Furthermore,

$$\lim_{\substack{h\to 0\\ h \text{ in } H_2}} \frac{f(g(x+h)) - f(g(x))}{h} = 0,$$

because h is of second kind, so $g(x+h) - g(x) = 0$, $g(x+h) = g(x)$ and therefore $f(g(x+h)) - f(g(x)) = 0$. The limit here is taken with h approaching 0, but h of the second kind.

On the other hand, if we take the limit with h of the first kind, then the original argument applies, i.e. we can divide and multiply by

$$k = g(x+h) - g(x),$$

and we find

$$\lim_{\substack{h\to 0\\ h \text{ in } H_1}} \frac{f(g(x+h)) - f(g(x))}{h} = f'(g(x))g'(x)$$

as before. But $g'(x) = 0$. Hence the limit is $0 = f'(g(x))g'(x)$ when h approaches 0, whether h is of first kind or second kind. This concludes the proof.

EXERCISES

Find the derivatives of the following functions.

1. $(x+1)^8$
2. $(2x-5)^{1/2}$
3. $(\sin x)^3$
4. $(\log x)^5$
5. $\sin 2x$
6. $\log(x^2+1)$
7. $e^{\cos x}$
8. $\log(e^x + \sin x)$
9. $\sin\left(\log x + \frac{1}{x}\right)$
10. $\frac{x+1}{\sin 2x}$
11. $(2x^2+3)^3$
12. $\cos(\sin 5x)$
13. $\log(\cos 2x)$
14. $\sin[(2x+5)^2]$
15. $\sin[\cos(x+1)]$
16. $\sin(e^x)$
17. $\frac{1}{(3x-1)^4}$
18. $\frac{1}{(4x)^3}$
19. $\frac{1}{(\sin 2x)^2}$
20. $\frac{1}{(\cos 2x)^2}$

21. $\frac{1}{\sin 3x}$
22. $(\sin x)(\cos x)$
23. $(x^2+1)e^x$
24. $(x^3+2x)(\sin 3x)$
25. $\frac{1}{\sin x + \cos x}$
26. $\frac{\sin 2x}{e^x}$
27. $\frac{\log x}{x^2+3}$
28. $\frac{x+1}{\cos 2x}$
29. $(2x-3)(e^x+x)$
30. $(x^3-1)(e^{3x}+5x)$
31. $\frac{x^3+1}{x-1}$
32. $\frac{x^2-1}{2x+3}$
33. $(x^{4/3}-e^x)(2x+1)$
34. $(\sin 3x)(x^{1/4}-1)$
35. $\sin(x^2+5x)$
36. e^{3x^2+8}
37. $\frac{1}{\log(x^4+1)}$
38. $\frac{1}{\log(x^{1/2}+2x)}$
39. $\frac{2x}{e^x}$
40. Relax.

SUPPLEMENTARY EXERCISES

Find the derivatives of the following functions.

1. $(2x+1)^2$
2. $(2x+5)^3$
3. $(5x+3)^7$
4. $(7x-2)^{81}$
5. $(2x^2+x-5)^3$
6. $(2x^3-3x)^4$
7. $(3x+1)^{1/2}$
8. $(2x-5)^{5/4}$
9. $(x^2+x-1)^{-2}$
10. $(x^4+5x+6)^{-1}$
11. $(x+5)^{-5/3}$
12. $(x^3+2x+1)^3$
13. $(x-1)(x-5)^3$
14. $(2x^2+1)^2(x^2+3x)$
15. $(x^3+x^2-2x-1)^4$
16. $(x^2+1)^3(2x+5)^2$
17. $\frac{(x+1)^{3/4}}{(x-1)^{1/2}}$
18. $\frac{(2x+1)^{1/2}}{(x+5)^5}$
19. $\frac{(2x^2+x-1)^{5/2}}{(3x+2)^9}$
20. $\frac{(x^2+1)(3x-7)^8}{(x^2+5x-4)^3}$
21. $\sqrt{2x+1}$
22. $\sqrt{x+3}$
23. $\sqrt{x^2+x+5}$
24. $\sqrt{2x^3-x+1}$

In the following exercises, we may assume that there are functions $\sin u$, $\cos u$, $\log u$, and e^u whose derivatives are given by the following formulas:

$$\frac{d\sin u}{du} = \cos u, \qquad \frac{d\cos u}{du} = -\sin u,$$

$$\frac{d(e^u)}{du} = e^u, \qquad \frac{d\log u}{du} = \frac{1}{u}.$$

Find the derivative of each function (with respect to x):

25. $\sin(x^3+1)$
26. $\cos(x^3+1)$
27. e^{x^3+1}
28. $\log(x^3+1)$
29. $\sin(\cos x)$
30. $\cos(\sin x)$
31. $e^{\sin(x^3+1)}$
32. $\log[\sin(x^3+1)]$
33. $\sin[(x+1)(x^2+2)]$
34. $\log(2x^2+3x+5)$
35. $e^{(x+1)(x-3)}$
36. e^{2x+1}
37. $\sin(2x+5)$
38. $\cos(7x+1)$
39. $\log(2x+1)$
40. $\log\frac{2x+1}{x+3}$
41. $\sin\frac{x-5}{2x+4}$
42. $\cos\frac{2x-1}{x+3}$
43. e^{2x^2+3x+1}
44. $\log(4x^3-2x)$
45. $\sin[\log(2x+1)]$
46. $\cos(e^{2x})$
47. $\cos(3x^2-2x+1)$
48. $\sin\left(\frac{x^2-1}{2x^3+1}\right)$
49. $(2x+1)^{80}$
50. $(\sin x)^{50}$
51. $(\log x)^{49}$
52. $(\sin 2x)^4$
53. $(e^{2x+1}-x)^5$
54. $(\log x)^{20}$
55. $(3\log(x^2+1) - x^3)^{1/2}$
56. $(\log(2x+3))^{4/3}$
57. $\frac{\sin 2x}{\cos 3x}$
58. $\frac{\sin(2x+5)}{\cos(x^2-1)}$
59. $\frac{\log 2x^2}{\sin x^3}$
60. $\frac{e^{x^3}}{x^2-1}$
61. $\frac{x^4+4}{\cos 2x}$
62. $\frac{\sin(x^3-2)}{\sin 2x}$
63. $\frac{(2x^2+1)^4}{(\cos x^3)}$
64. $\frac{e^{-x}}{\cos 2x}$
65. e^{-3x}
66. e^{-x^2}
67. e^{-4x^2+x}
68. $\sqrt{e^x+1}$
69. $\frac{\log(x^2+2)}{e^{-x}}$
70. $\frac{\log(2x+1)}{\sin(4x+5)}$

§7. HIGHER DERIVATIVES

Given a differentiable function f defined on an interval, its derivative f' is also a function on this interval. If it turns out to be also differentiable (this being usually the case), then its derivative is called the **second derivative** of f

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Math 2107

Find the derivatives of each of the following functions

$f(x) = 2x^{1/3}$

$f(x) = 2x^{-1/3}$

$f(x) = \frac{2}{x^{1/3}}$

$f(x) = \frac{2}{x^{-1/3}}$

$f(x) = \frac{1}{2x^{1/3}}$

$f(x) = \frac{1}{2x^{-1/3}}$

$f(x) = \frac{1}{3}x^4 - 5x^3 + x^2 - 2$

$f(x) = 4x^5 - 7x^3 + 2x - 1 - 3x^2 - x^7$

$f(x) = \pi x^7 + 7x^\pi$

$f(x) = 2^4$

$f(x) = x^{\sqrt{2}}$

$f(x) = x^3 \cdot (x^5 + x^2 + 2)$

$f(x) = (x^3 + x) \cdot (x - 1)$

$f(x) = (2x^4 - 1) \cdot (x^2 + 1)$

$f(x) = (x + 1) \cdot (x^2 + 5x^{3/2})$

$$f(x) = (2x-5)\cdot(3x^4+5x+2)$$

$$f(x) = \left(x^{-2/3} + x^2\right)\cdot\left(x^3 + \frac{1}{x}\right)$$

$$f(x) = (2x+3)\cdot\left(\frac{1}{x^2} + \frac{1}{x}\right)$$

$$f(x) = \frac{2x+1}{3x+5}$$

$$f(x) = \frac{5x}{x^7+3x+1}$$

$$f(x) = \frac{x^2+2x-2}{(x+1)\cdot(x+2)}$$

$$f(x) = \frac{x^{-5/4}}{x^2+x-1}$$

$$f(x) = \frac{x}{x+5}$$

$$f(x) = \frac{x+5}{x}$$

$$f(x) = \frac{x^2}{x^2+1}$$

$$f(x) = \frac{x^2+1}{x^2}$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{1}{x^{10}+2}$$

$$f(x) = \frac{x^2+2x+7}{8x}$$

$$f(x) = \frac{2x+1}{x^2+x-4}$$

$$f(x) = \frac{x^2-x}{x^2+1}$$

$$f(x) = (2x+1)^2$$

$$f(x) = (5x+7)^3$$

$$f(x) = (3x+1)^{1/2}$$

$$f(x) = (x^2+x-1)^{-2}$$

$$f(x) = (x^3+x^2-2x-1)^4 \cdot (x^2+5)^{-4/7}$$

$$f(x) = \frac{(x+1)^{3/4}}{(x-1)^{1/2}}$$

$$f(x) = \frac{(x^2+2x+1)^4}{(x^4-3x^2+x-5)^3}$$

$$f(x) = (2x^3+1)^2 \cdot (x^2+3x)$$

$$f(x) = \frac{(x^2+1)\cdot(3x-7)^8}{(x^5+5x-4)^3}$$

$$f(x) = \sin x$$

$$f(x) = \cos x$$

$$f(x) = e^x$$

$$f(x) = \frac{\sin x}{\cos x}$$

$$f(x) = (\sin x)^2$$

$$f(x) = \frac{\cos x}{\sin x}$$

$$f(x) = \frac{1}{\sin x}$$

$$f(x) = \frac{1}{\cos x}$$

$$f(x) = \sqrt{e^x + 1}$$

$$f(x) = (4x^2)^6$$

$$f(x) = \frac{1}{(4x)^6}$$

$$f(x) = \frac{1}{4 \cdot x^6}$$

$$f(x) = \sin(x^3+1)$$

$$f(x) = e^{\sin(x^3+1)}$$

$$f(x) = [\sin(x^3+1)]^2$$

$$f(x) = e^{[\sin(x^3+1)]^2}$$

$$f(x) = e^{4x^2+3x+1}$$

$$f(x) = \cos(e^{2x})$$

$$f(x) = [\cos(e^{2x})]^4$$

$$f(x) = \sin x \cdot \cos x$$

$$f(x) = \sin(\cos x)$$

$$f(x) = \sin[\cos(x^2)]$$

$$f(x) = (\sin[\cos(x^2)])^3$$

$$f(x) = (\sin x)^{50}$$

$$f(x) = \sin(x^{50})$$

$$f(x) = \sin(50x)$$

$$f(x) = [\sin(50x)]^4$$

$$f(x) = [\ln(x^2+3)]^{4/3}$$

$$f(x) = \frac{e^{-4x^2+x}}{\sin(2x)}$$

$$f(x) = \sqrt{e^{x^2} + x^5}$$

$$f(x) = \frac{\ln(x^2+2)}{e^{-x}}$$

$$f(x) = (\sin[\ln(x^4)])^4$$

$$f(x) = \ln(x^{20})$$

$$f(x) = [\ln(x)]^{20}$$

$f(x) = \ln(x^4 + x^2 + 2)$

$f(x) = \ln(x^2 + 1)$

$f(x) = \sin[\ln(x^2+1)]$

$f(x) = e^{\sin[\ln(x^2+1)]}$

$f(x) = \left(e^{\sin[\ln(x^2+1)]}\right)^5$

$f(x) = x^2$

$f(x) = 2^x$

$f(x) = e^{x^3}$

$f(x) = 3^{x^3}$

$f(x) = x^x$

$f(x) = 2^{2}$

$f(x) = 2^{x^3+x+1}$