Math 2107

Def. $f(x) := e^x := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$\Downarrow$

$$f'(x) = 0 + 1 + \frac{2x}{2\cdot 1} + \frac{3x^2}{3\cdot 2!} + \frac{4x^3}{4\cdot 3!} + \frac{5x^4}{5\cdot 4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$=: e^x =: f(x)$$

---

i.e., $\frac{de^x}{dx} = e^x$, so $\frac{de^u}{dx} = \frac{de^u}{du}\cdot\frac{du}{dx} = e^u \frac{du}{dx}$

i.e. $\left[e^{J(x)}\right]' = e^{J(x)} \cdot J'(x)$

" $f(x) = e^{4x^2+3x+1} \Rightarrow f'(x) = e^{4x^2+3x+1} \cdot (8x+3)$

$f(x) = \cos(e^{2x}) \Rightarrow f'(x) = -\sin(e^{2x}) \cdot e^{2x} \cdot 2$

$$= -2\cdot e^{2x} \cdot \sin(e^{2x})$$

$f(x) = \left[\cos(e^{2x})\right]^4$

$\Downarrow$

$f'(x) = 4 \cdot \left[\cos(e^{2x})\right]^3 \cdot \left[-\sin(e^{2x}) \cdot e^{2x} \cdot 2\right]$

$$= -8 \cdot e^{2x} \cdot \sin(e^{2x}) \cdot \cos^3(e^{2x}).$$

$f(x) = \sin x \cdot \cos x$

$\Downarrow$

$f'(x) = \sin x \cdot [-\sin x] + \cos x \cdot [\cos x]$

$$= \cos^2 x - \sin^2 x \neq \tfrac{1}{2}\cdot 2 \cdot \sin x \cdot \cos x = \tfrac{1}{2}\sin(2x)$$

$\boxed{\text{OR}}$ $f(x) = \sin x \cdot \cos x = \frac{1}{2} \cdot 2 \cdot \sin x \cdot \cos x = \frac{1}{2} \cdot \sin(2x)$

$\Downarrow$

$f'(x) = \frac{1}{2} \cdot \cos(2x) \cdot 2 = \cos(2x) = \cos^2(x) - \sin^2(x).$

Note: Recall that

$$\sin(A+B) = \sin(A)\cdot\cos(B) + \cos(A)\cdot\sin(B)$$

So, upon taking $A = B = x$, we have that

$$\sin(2x) = \sin(x+x) = \sin x\cdot\cos x + \cos x\cdot\sin x$$
$$= 2\sin x\cdot\cos x$$

Also,

$$\cos(A+B) = \cos(A)\cdot\cos(B) - \sin(A)\cdot\sin(B)$$

So, upon taking $A = B = x$, we have that

$$\cos(2x) = \cos(x+x) = \cos x\cdot\cos x - \sin x\cdot\sin x$$
$$= \cos^2 x - \sin^2 x$$

Further, since $\sin^2 x + \cos^2 x = 1$

we have that

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= 1-\sin^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

and also that

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1-\cos^2 x)$$
$$= 2\cos^2 x - 1$$

which leads to the $\frac{1}{2}$-angle formulas

$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \& \quad \sin^2 x = \frac{1-\cos(2x)}{2}$$

i.e., $$\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos(x)}{2} \quad \& \quad \sin^2\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{2}.$$

Note: Be sure not to confuse

$f(x) = \sin x \cdot \cos x = \cos x \cdot \sin x$

with either

$f(x) = \sin[\cos x]$ or $f(x) = \cos[\sin x]$

Indeed

$$f(x) = \sin x \cdot \cos x$$
$$\Downarrow$$
$$\begin{aligned} f'(x) &= \sin x \cdot (\cos x)' + \cos x \cdot (\sin x)' \\ &= \sin x \cdot (-\sin x) + \cos x \cdot \cos x \\ &= \cos^2 x - \sin^2 x, \end{aligned}$$

whereas as we saw above,

$$f(x) = \sin[\cos x]$$
$$\Downarrow$$
$$\begin{aligned} f'(x) &= \cos[\cos x] \cdot (\cos x)' \\ &= \cos[\cos x] \cdot (-\sin x) \\ &= -\sin x \cdot \cos[\cos x] \end{aligned}$$

while $f(x) = \cos[\sin x]$

$$\Downarrow$$
$$\begin{aligned} f'(x) &= -\sin[\sin x] \cdot (\sin x)' \\ &= -\sin[\sin x] \cdot \cos x \\ &= -\cos x \cdot \sin[\sin x]. \end{aligned}$$

Indeed, one has

$$(f\cdot g)'(x) = \big(f(x)\cdot g(x)\big)'$$
$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

versus

$$(f\circ g)'(x) = f'[g(x)]\cdot g'(x).$$

Recall: $(f\cdot g)(x) := f(x)\cdot g(x)$
= the pointwise product of $f(x)$ & $g(x)$

whereas

$$f\circ g\,(x) := f[g(x)]$$

i.e., $f\circ g$ = the composite of the <u>inner function</u> g followed by the <u>outer function</u> f

i.e., $f\cdot g$ = the pointwise product of f & g

while $f\circ g$ = the composition product of f & g

Junk of x

$f(x) = \sin[J(x)]$

$\Downarrow$

$f'(x) = \cos[J(x)] \cdot J'(x)$

$f(x) = \cos[J(x)]$

$\Downarrow$

$f'(x) = -\sin[J(x)] \cdot J'(x)$

$\boxed{OR}$

$y = f(x) = \sin(u)$

$\Downarrow$

$$\frac{dy}{dx} = f'(x) = \cos(u) \cdot \frac{du}{dx}$$

$y = f(x) = \cos(u)$

$\Downarrow$

$$\frac{dy}{dx} = f'(x) = -\sin(u)\frac{du}{dx}$$

$\boxed{OR}$

$y = \sin(u)$

$\Downarrow$

$$\frac{dy}{dx} = \frac{d\sin(u)}{dx}$$

$$= \frac{d\sin(u)}{du} \cdot \frac{du}{dx}$$

$$= \cos(u) \cdot \frac{du}{dx}$$

$y = \cos(u)$

$\Downarrow$

$$\frac{dy}{dx} = \frac{d\cos(u)}{dx}$$

$$= \frac{d\cos(u)}{du} \cdot \frac{du}{dx}$$

$$= -\sin(u)\frac{du}{dx}$$

$$y = f(x) = \tan[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = \sec^2[J(x)] \cdot J'(x)$$

$\boxed{\text{OR}}$ $\quad y = \tan(u)$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d\tan(u)}{dx} = \frac{d\tan(u)}{du}\,\frac{du}{dx}$$

$$= \sec^2(u)\cdot\frac{du}{dx}$$

$$y = f(x) = \cot[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = -\csc^2[J(x)] \cdot J'(x)$$

$\boxed{\text{OR}}$ $\quad y = \cot(u)$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d\cot(u)}{dx} = \frac{d\cot(u)}{du}\cdot\frac{du}{dx}$$

$$= -\csc^2(u)\cdot\frac{du}{dx}$$

$$y = f(x) = \sec[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = \sec[J(x)] \cdot \tan[J(x)] \cdot J'(x)$$

$\boxed{\text{OR}}$

$$y = \sec(u)$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d \sec(u)}{dx} = \frac{d \sec(u)}{du} \cdot \frac{du}{dx}$$

$$= \sec(u) \cdot \tan(u) \cdot \frac{du}{dx}$$

$$y = f(x) = \csc[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = -\csc[J(x)] \cdot \cot[J(x)] \cdot J'(x)$$

$\boxed{\text{OR}}$

$$y = \csc(u)$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d \csc(u)}{dx} = \frac{d \csc(u)}{du} \cdot \frac{du}{dx}$$

$$= -\csc(u) \cdot \cot(u) \cdot \frac{du}{dx}$$

$$y = f(x) = e^{J(x)}$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = e^{J(x)} \cdot J'(x)$$

$\boxed{\text{OR}}$

$$y = e^{u}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d e^{u}}{dx} = \frac{d e^{u}}{du} \cdot \frac{du}{dx}$$

$$= e^{u} \cdot \frac{du}{dx}$$

$$y = \ln[J(x)] = \log_e[J(x)] \quad , \text{ for } J(x) > 0$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = \frac{1}{J(x)} \cdot J'(x) = \frac{J'(x)}{J(x)}$$

$\boxed{\text{OR}}$

$$y = \ln(u) \quad \text{for } u > 0$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d \ln(u)}{dx} = \frac{d \ln(u)}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot \frac{du}{dx}$$

9

$y = f(x) = [J(x)]^c$, where c = a constant

$\Downarrow$

$\frac{dy}{dx} = f'(x) = c \cdot [J(x)]^{c-1} \cdot J'(x)$

$\boxed{OR}$

$y = u^c$

c: a constant

u: a variable = a function of x

$\Downarrow$

$$\frac{dy}{dx} = \frac{du^c}{dx} = \frac{du^c}{du} \cdot \frac{du}{dx}$$

$$= c \cdot u^{c-1} \cdot \frac{du}{dx}$$

$$f(x) = \sin\left[\cos(x^2)\right] = \sin\left[J(x)\right]$$

$$\Downarrow$$

$$f'(x) = \cos\left[J(x)\right] \cdot J'(x)$$

or

$$f(x) = \sin(u)$$

$$\Downarrow$$

$$f'(x) = \cos(u) \cdot \frac{du}{dx}$$

So:

$$f(x) = \sin\left[\cos(x^2)\right]$$

$$\Downarrow$$

$$f'(x) = \cos\left[\cos(x^2)\right] \cdot \left[-\sin(x^2) \cdot (2x)\right]$$

OR

$$y = \sin\left[\cos(x^2)\right]$$

$$\frac{dy}{dx} = \frac{d \sin\left[\cos(x^2)\right]}{dx}$$

$$= \frac{d \sin\left[\cos(x^2)\right]}{d \cos(x^2)} \cdot \frac{d \cos(x^2)}{dx}$$

$$= \cos\left[\cos(x^2)\right] \cdot \frac{d \cos(x^2)}{dx^2} \cdot \frac{dx^2}{dx}$$

$$= \cos\left[\cos(x^2)\right] \cdot \left[-\sin(x^2) \cdot 2x\right]$$

(11)

$$f(x) = \left(\sin\left[\cos(x^2)\right]\right)^3 = \left(J(x)\right)^3$$

$$\Downarrow$$

$$f'(x) = 3\cdot\left(J(x)\right)^2\cdot J'(x)$$

$$= 3\cdot\left(\sin\left[\cos(x^2)\right]\right)^2\cdot\left[\cos\left[\cos(x^2)\right]\cdot\left(-\sin(x^2)\cdot 2x\right)\right]$$

$\boxed{\text{OR}}$

$y = f(x) = u^3$ where $u = \sin[\cos(x^2)]$

$= \sin(v)$ where

$v = \cos(x^2)$

$= \cos(w)$

where

$w = x^2$

$$\Downarrow$$

$$f'(x) = \frac{dy}{dx} = \frac{d\,u^3}{dx} = \frac{du^3}{du}\cdot\frac{du}{dx} = 3u^2\cdot\frac{du}{dx}$$

$$= 3u^2\cdot\frac{d\sin(v)}{dx} = 3u^2\cdot\frac{d\sin(v)}{dv}\cdot\frac{dv}{dx}$$

$$= 3u^2\cdot\cos(v)\cdot\frac{dv}{dx} = 3u^2\cdot\cos(v)\cdot\frac{d\cos(w)}{dx}$$

$$= 3u^2\cdot\cos(v)\cdot\frac{d\cos(w)}{dw}\cdot\frac{dw}{dx}$$

$$= 3u^2\cdot\cos(v)\cdot\left(-\sin(w)\right)\cdot\frac{dw}{dx}$$

$$= 3u^2\cdot\cos(v)\cdot\left(-\sin(w)\right)\cdot\frac{dx^2}{dx}$$

$$= 3u^2\cdot\cos(v)\cdot\left(-\sin(w)\right)\cdot 2x$$

$$= 3\left(\sin\left[\cos(x^2)\right]\right)^2\cdot\cos\left[\cos(x^2)\right]\cdot\left(-\sin(x^2)\right)\cdot 2x$$

$$= -6x\cdot\left(\sin\left[\cos(x^2)\right]\right)^2\cdot\cos\left[\cos(x^2)\right]\cdot\sin(x^2)$$

(12)

$$f(x) = x^7 + x^5 \Rightarrow f'(x) = 7x^6 + 5x^4$$

$$f(x) = -6x^3 + 12x^2 - 4x + 7$$
$$\Downarrow$$
$$f'(x) = -18x^2 + 24x - 4$$

$$f(x) = \frac{1}{10}\cdot x^{10} + \frac{1}{9}\cdot x^9 + \frac{1}{8}\cdot x^8$$
$$\Downarrow$$
$$f'(x) = x^9 + x^8 + x^7$$

$$f(x) = (3x+5)^2 = 9x^2 + 30x + 25$$
$$\Downarrow$$
$$f'(x) = 18x + 30$$

$\boxed{\text{OR}}$ $f(x) = (3x+5)^2 \Rightarrow f'(x) = 2\cdot(3x+5)^1\cdot 3$
$= 6\cdot(3x+5) = 18x + 30$

$$f(x) = (-2x^2+1)^3 = (-2x^2+1)\cdot(-2x^2+1)^2$$
$$= (-2x^2+1)\cdot(4x^4 - 4x^2 + 1)$$
$$= -8x^6 + 8x^4 - 2x^2 + 4x^4 - 4x^2 + 1$$
$$= -8x^6 + 12x^4 - 6x^2 + 1$$
$$\Downarrow$$
$$f'(x) = -48x^5 + 48x^3 - 12x$$

$\boxed{\text{OR}}$ $f(x) = (-2x^2+1)^3 = (a+b)^3$ where $a = -2x^2$ & $b = 1$
$= a^3 + 3a^2b + 3ab^2 + b^3$
$= a^3 + 3a^2 + 3a + 1$
$= (-2x^2)^3 + 3\cdot(-2x^2)^2 + 3\cdot(-2x^2) + 1$
$= -8x^6 + 3\cdot 4x^4 - 6x^2 = -8x^6 + 12x^4 - 6x^2 + 1$
$\Downarrow$
$f'(x) = -48x^5 + 48x^3 - 12x$

$\boxed{\text{OR}}$ $f(x) = (-2x^2+1)^3 \Rightarrow f'(x) = 3\cdot(-2x^2+1)^2\cdot(-4x)$
$= -12x\cdot(4x^4 - 4x^2 + 1)$
$= -48x^5 + 48x^3 - 12x$

$$f(x) = (3x^2 - 7x + 1)(x^2 + x - 1)$$

$$= 3x^2\cdot(x^2 + x - 1) - 7x\cdot(x^2 + x - 1) + 1\cdot(x^2 + x - 1)$$

$$= 3x^4 + 3x^3 - 3x^2 - 7x^3 - 7x^2 + 7x + x^2 + x - 1$$

$$= 3x^4 - 4x^3 - 9x^2 + 8x - 1$$

$\boxed{\text{OR}}$

$$\begin{array}{r} 3x^2 - 7x + 1 \\ x^2 + x - 1 \\ \hline 3x^4 - 7x^3 + x^2 \qquad\qquad \\ 3x^3 - 7x^2 + x \qquad \\ -3x^2 + 7x - 1 \\ \hline \end{array}$$

so $f(x) = 3x^4 - 4x^3 - 9x^2 + 8x - 1$

$$\Downarrow$$

$$f'(x) = 12x^3 - 12x^2 - 18x + 8$$

$\boxed{\text{OR}}$ $f(x) = (3x^2 - 7x + 1)\cdot(x^2 + x - 1)$

$$\Downarrow$$

$$f'(x) = (3x^2 - 7x + 1)\cdot(2x + 1) + (x^2 + x - 1)\cdot(6x - 7)$$

$$= 6x^3 + 3x^2 - 14x^2 - 7x + 2x + 1$$
$$+ 6x^3 - 7x^2 + 6x^2 - 7x - 6x + 7$$

$$= 12x^3 - 4x^2 - 8x^2 - 14x - 4x + 8$$

$$= 12x^3 - 12x^2 - 18x + 8$$

(14)

$$f(x) = (x^2+1)\cdot(x^3+1)$$
$$= x^2\cdot(x^3+1) + 1\cdot(x^3+1)$$
$$= x^5 + x^2 + x^3 + 1$$
$$= x^5 + x^3 + x^2 + 1$$

$$\Downarrow$$

$$f'(x) = 5x^4 + 3x^2 + 2x$$

$$\begin{array}{r} x^3+1 \\ x^2+1 \\ \hline x^5 + \quad x^2 \\ x^3 + \quad 1 \\ \hline x^5+x^3+x^2+1 \end{array}$$

$\boxed{OR}$ $$f(x) = (x^2+1)\cdot(x^3+1)$$

$$\Downarrow$$

$$f'(x) = (x^2+1)\cdot 3x^2 + (x^3+1)\cdot 2x$$
$$= \underbrace{3x^4 + 3x^2 + 2x^4} + 2x$$
$$= 5x^4 + 3x^2 + 2x$$

$$f(x) = (x-1)\cdot(x^5+x^4+x^3+x^2+x+1)$$
$$= x^6 + x^5 + x^4 + x^3 + x^2 + x - x^5 - x^4 - x^3 - x^2 - x - 1$$
$$= x^6 - 1$$

$$\Downarrow$$

$$f'(x) = 6x^5$$

$\boxed{OR}$

$$\begin{array}{r} x^5+x^4+x^3+x^2+x+1 \\ x-1 \\ \hline x^6+x^5+x^4+x^3+x^2+x \\ -x^5-x^4-x^3-x^2-x-1 \\ \hline x^6 \qquad\qquad\qquad\qquad -1 \end{array}$$

$\boxed{OR}$ $$f(x) = (x-1)\cdot(x^5+x^4+x^3+x^2+x+1)$$

$$\Downarrow$$

$$f'(x) = (x-1)\cdot(5x^4+4x^3+3x^2+2x+1) + (x^5+x^4+x^3+x^2+x+1)\cdot 1$$
$$= 5x^5 + 4x^4 + 3x^3 + 2x^2 + x + x^5 + x^4 + x^3 + x^2 + x + 1$$
$$- 5x^4 - 4x^3 - 3x^2 - 2x - 1$$
$$= 6x^5 + (1+4-5)x^4 + (1+3-4)x^3 + (1+2-3)x^2 + (1+1-2)x + 1 - 1$$
$$= 6x^5$$

$$f(x) = \frac{5x+6}{x^2+1}$$

$$\Downarrow$$

$$f'(x) = \frac{(x^2+1)\cdot 5 - (5x+6)\cdot 2x}{(x^2+1)^2}$$

$$= \frac{5x^2+5-10x^2-12x}{(x^2+1)^2} = \frac{-5x^2-12x+5}{(x^2+1)^2}$$

$$f(x) = \frac{x^3-x}{x^5-x^3+1}$$

$$\Downarrow$$

$$f'(x) = \frac{(x^5-x^3+1)\cdot(3x^2-1) - (x^3-x)\cdot(5x^4-3x^2)}{(x^5-x^3+1)^2}$$

$$= \frac{3x^7-x^5-3x^5+x^3+3x^2-1-(5x^7-3x^5-5x^5+3x^3)}{(x^5-x^3+1)^2}$$

$$= \frac{(3-5)x^7+(3+5-1-3)x^5+(1-3)x^3+3x^2-1}{(x^5-x^3+1)^2}$$

$$= \frac{-2x^7+4x^5-2x^3+3x^2-1}{(x^5-x^3+1)^2}$$

$$f(x) = \frac{x^2}{2} = \frac{1}{2}\cdot x^2$$

$$\Downarrow$$

$$f'(x) = \frac{1}{2}\cdot 2\cdot x^1 = x$$

$\boxed{\text{OR}}$

$$f(x) = \frac{x^2}{2}$$

$$\Downarrow$$

$$f'(x) = \frac{2\cdot(2x) - x^2\cdot 0}{2^2} = \frac{4x}{4} = x$$

$$f(x) = (x^3+1)(x^2+1)\cdot(x+1)$$

$$= (x^3+1)\cdot(x^3+x^2+x+1)$$

$$= x^6 + x^5 + x^4 + x^3 + x^3 + x^2 + x + 1$$

$$= x^6 + x^5 + x^4 + 2x^3 + x^2 + x + 1$$

$$\Downarrow$$

$$f'(x) = 6x^5 + 5x^4 + 4x^3 + 6x^2 + 2x + 1$$

$\boxed{\text{OR}}$ $f(x) = (x^3+1)\cdot(x^2+1)\cdot(x+1) = u\cdot v\cdot w$

$$\Downarrow$$

$$f'(x) = u'\cdot v\cdot w + u\cdot v'\cdot w + u\cdot v\cdot w'$$

$$= 3x^2\cdot(x^2+1)\cdot(x+1) + (x^3+1)\cdot 2x\cdot(x+1)$$
$$+ (x^3+1)\cdot(x^2+1)\cdot 1$$

$$= 3x^2\cdot(x^3+x^2+x+1) + (2x^4+2x)(x+1) + x^5+x^3+x^2+1$$

$$= 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x^5 + 2x^4 + 2x^2 + 2x + x^5 + x^3 + x^2 + 1$$

$$= 6x^5 + 5x^4 + 4x^3 + 6x^2 + 2x + 1$$

$$f(x) = \frac{(x^3 - 2x^2 + 7x + 1)\cdot(x^2 - x + 3)}{x^3+1}$$

Now,

$$\begin{array}{rrrrrr}
 & x^3 & -2x^2 & +7x & +1 & \\
 & & x^2 & -x & +3 & \\
\hline
x^5 & -2x^4 & +7x^3 & +x^2 & & \\
 & -x^4 & +2x^3 & -7x^2 & -x & \\
 & & 3x^3 & -6x^2 & +21x & +3 \\
\hline
x^5 & -3x^4 & +12x^3 & -12x^2 & +20x & +3
\end{array}$$

Hence,

$$f(x) = \frac{x^5 - 3x^4 + 12x^3 - 12x^2 + 20x + 3}{x^3+1}$$

$$\Downarrow$$

$$f'(x) = \frac{(x^3+1)\cdot(5x^4 - 12x^3 + 36x^2 - 24x + 20) - (x^5 - 3x^4 + 12x^3 - 12x^2 + 20x + 3)\cdot 3x^2}{(x^3+1)^2}$$

$$= \frac{\begin{array}{l} 5x^7 - 12x^6 + 36x^5 - 24x^4 + 20x^3 \\ \qquad + 5x^4 - 12x^3 + 36x^2 - 24x + 20 \\ -3x^7 + 9x^6 - 36x^5 + 36x^4 - 60x^3 - 9x^2 \end{array}}{(x^3+1)^2}$$

$$= \frac{2x^7 - 3x^6 + 17x^4 - 52x^3 + 27x^2 - 24x + 20}{(x^3+1)^2}$$

$$f(x) = \frac{(x^2+1)\cdot(x^4+1)}{(x^3+1)\cdot(x^5+1)} = \frac{(x^4+1)\cdot(x^2+1)}{(x^5+1)\cdot(x^3+1)} = \frac{x^6+x^4+x^2+1}{x^8+x^5+x^3+1}$$

$$\Downarrow$$

$$f'(x) = \frac{(x^8+x^5+x^3+1)\cdot(6x^5+4x^3+2x) - (x^6+x^4+x^2+1)\cdot(8x^7+5x^4+3x^2)}{(x^8+x^5+x^3+1)^2}$$

Now,

$$\begin{array}{r}
6x^5+4x^3+2x \\
x^8+x^5+x^3+1 \\
\hline
6x^{13} + 4x^{11} + 2x^9 \\
6x^{10} + 4x^8 + 2x^6 \\
6x^8 + 4x^6 + 2x^4 \\
6x^5 + 4x^3 + 2x \\
\hline
6x^{13} + 4x^{11} + 6x^{10} + 2x^9 + 10x^8 + 6x^6 + 6x^5 + 2x^4 + 4x^3 + 2x
\end{array}$$

while

$$\begin{array}{r}
8x^7 + 5x^4 + 3x^2 \\
x^6 + x^4 + x^2 + 1 \\
\hline
8x^{13} + 5x^{10} + 3x^8 \\
8x^{11} + 5x^8 + 3x^6 \\
8x^9 + 5x^6 + 3x^4 \\
8x^7 + 5x^4 + 3x^2 \\
\hline
8x^{13} + 8x^{11} + 5x^{10} + 8x^9 + 8x^8 + 8x^7 + 8x^6 + 8x^4 + 3x^2
\end{array}$$

Hence,

$$f'(x) = \frac{(6-8)x^{13} + (4-8)x^{11} + (6-5)x^{10} + (2-8)x^9 + (10-8)x^8 - 8x^7 + (6-8)x^6 + 6x^5 + (2-8)x^4 + 4x^3 - 3x^2 + 2x}{(x^8+x^5+x^3+1)^2}$$

$$= \frac{-2x^{13} - 4x^{11} + x^{10} - 6x^9 + 2x^8 - 8x^7 - 2x^6 + 6x^5 - 6x^4 + 4x^3 - 3x^2 + 2x}{(x^8+x^5+x^3+1)^2}$$

$\boxed{\text{OR}}$

$$f(x) = \frac{(x^2+1)\cdot(x^4+1)}{(x^3+1)\cdot(x^5+1)}$$

$$\Downarrow$$

$$f'(x) = \frac{(x^3+1)\cdot(x^5+1)\cdot\left[(x^2+1)\cdot 4x^3 + (x^4+1)\cdot 2x\right] - (x^2+1)\cdot(x^4+1)\cdot\left[(x^3+1)\cdot 5x^4 + (x^5+1)\cdot 3x^2\right]}{(x^3+1)^2\cdot(x^5+1)^2}$$

$$= \frac{(x^8+x^3+x^5+1)\cdot\left[4x^5+4x^3+2x^5+2x\right] - (x^6+x^2+x^4+1)\cdot\left[5x^7+5x^4+3x^7+3x^2\right]}{(x^3+1)^2\cdot(x^5+1)^2}$$

$$= \frac{(x^8+x^5+x^3+1)\cdot(6x^5+4x^3+2x) - (x^6+x^4+x^2+1)\cdot(8x^7+5x^4+3x^2)}{(x^3+1)^2\cdot(x^5+1)^2}$$

Now,

$$\begin{array}{r}
6x^5 + 4x^3 + 2x \\
x^8 + x^5 + x^3 + 1 \\
\hline
6x^{13} + 4x^{11} + 2x^9 \\
6x^{10} + 4x^8 + 2x^6 \\
6x^8 + 4x^6 + 2x^4 \\
6x^5 + 4x^3 + 2x \\
\hline
6x^{13} + 4x^{11} + 6x^{10} + 2x^9 + 10x^8 + 6x^6 + 6x^5 + 2x^4 + 4x^3 + 2x
\end{array}$$

while

$$\begin{array}{r}
8x^7 + 5x^4 + 3x^2 \\
x^6 + x^4 + x^2 + 1 \\
\hline
8x^{13} + 5x^{10} + 3x^8 \\
8x^{11} + 5x^8 + 3x^6 \\
8x^9 + 5x^6 + 3x^4 \\
8x^7 + 5x^4 + 3x^2 \\
\hline
8x^{13} + 8x^{11} + 5x^{10} + 8x^9 + 8x^8 + 8x^7 + 8x^6 + 8x^4 + 3x^2
\end{array}$$

Hence, as before,

$$f'(x) = \frac{-2x^{13} - 4x^{11} + x^{10} - 6x^9 + 2x^8 - 8x^7 - 2x^6 + 6x^5 - 6x^4 + 4x^3 - 3x^2 + 2x}{(x^3+1)^2\cdot(x^5+1)^2}$$

$$f(x) = [g(x)]^2 = g(x) \cdot g(x)$$

$$\Downarrow$$

$$f'(x) = g(x) \cdot g'(x) + g(x) \cdot g'(x)$$

$$= 2\, g(x) \cdot g'(x)$$

$$\Downarrow$$

$$f'(1) = 2 \cdot g(1) \cdot g'(1)$$

Note: $f(1) = [g(1)]^2$

But $f'(1) \neq [f(1)]' = 0$

Since $f(1) = [g(1)]^2 = $ a constant!!!

In otherwords, to compute $f'(1)$ you must differentiate 1st & then plug-in, as opposed to plugging in (& getting a constant) & then differentiating.

I hope none of you thought that

$$f'(1) = [f(1)]'$$

which is false. For example, if $g(x) = x^3 + 1$, then

$f(x) = g(x) \cdot g(x) = (x^3+1)^2 = x^6 + 2x^3 + 1$

so $f'(x) = 6x^5 + 6x^2$ $\boxed{\text{OR}}$ $f'(x) = 2 \cdot (x^3+1) \cdot 3x^2$
$= 6x^2 \cdot (x^3+1) = 6x^5 + 6x^2$

so $f'(1) = 6 \cdot 1^5 + 6 \cdot 1^2 = 6 \cdot 1 + 6 \cdot 1 = 6 + 6 = 12$
whereas $f(1) = (1^3+1)^2 = (1+1)^2 = 2^2 = 4$ so $(f(1))' = 4' = 0$!!!!

$$y = (x^2+2x+3)^2$$

$$\Downarrow$$

$$\frac{dy}{dx} = 2\cdot(x^2+2x+3)^1\cdot(2x+2) = 2(x^2+2x+3)\cdot 2\cdot(x+1)$$

$$= 4\cdot(x+1)\cdot(x^2+2x+3)$$

$\boxed{\text{OR}}$ $\quad y = (x^2+2x+3)^2 = ?$

Now,

$$\begin{array}{llllll}
 & a+b+c & & & & \\
 & a+b+c & & & & \\
\hline
a^2 & & + ab & & + ac & \\
 & b^2 & + & ab & & + bc \\
 & & +c^2 & & ac & + bc \\
\hline
\multicolumn{6}{l}{a^2+b^2+c^2+2(ab+ac+bc)}
\end{array}$$

abc

Hence $(a+b+c)^2 = a^2+b^2+c^2+2(ab+ac+bc)$

So

$$y = (x^2+2x+3)^2 = x^4+4x^2+9+2\cdot(2x^3+3x^2+6x)$$

$$= x^4+4x^2+9+4x^3+6x^2+12x$$

$$= x^4+4x^3+10x^2+12x+9$$

Note:

$$\begin{array}{r}
x^2+2x+3 \\
x^2+2x+3 \\
\hline
x^4+2x^3+3x^2 \qquad\qquad \\
2x^3+4x^2+6x \qquad \\
3x^2+6x+9 \\
\hline
x^4+4x^3+10x^2+12x+9
\end{array}$$

Hence

$$y = (x^2+2x+3)^2 = x^4+4x^3+10x^2+12x+9$$

$$\Downarrow$$

$$\frac{dy}{dx} = 4x^3+12x^2+20x+12$$

The question is, therefore,

does $4x^3+12x^2+20x+12 \stackrel{?}{=} 4(x+1)\cdot(x^2+2x+3)$?

If so, -1 is a root of $4x^3+12x^2+20x+12$

so

$$\begin{array}{r|rrrr} -1 & 4 & 12 & 20 & 12 \\ & & -4 & -8 & -12 \\ \hline & 4 & 8 & 12 & \boxed{0} \end{array} \leftarrow \text{BINGO!}$$

$$\downarrow$$

$$4x^2+8x^1+12\cdot x^0$$

so $$4x^3+12x^2+20x+12 = [x-(-1)]\cdot[4x^2+8x+12]$$
$$= (x+1)\cdot 4\cdot[x^2+2x+3]$$
$$= 4\cdot(x+1)\cdot(x^2+2x+3)$$

Note: One also has that

$$\begin{array}{r} x^2+2x+3 \\ 4x+4 \\ \hline 4x^3+8x^2+12x \quad \\ 4x^2+8x+12 \\ \hline 4x^3+12x^2+20x+12 \end{array}$$

!!!

$$y = \left(\frac{x^3}{3}+1\right)^5 + \left(\frac{x^2}{2}+1\right)^4 + 3$$

$$\Downarrow$$

$$\frac{dy}{dx} = 5\cdot\left(\frac{x^3}{3}+1\right)^4\cdot\left(x^2\right) + 4\cdot\left(\frac{x^2}{2}+1\right)^3\cdot x$$

$$= 5x^2\cdot\left(\frac{x^3}{3}+1\right)^4 + 4x\cdot\left(\frac{x^2}{2}+1\right)^3$$

$$y = \left[(x^2+1)^{10}+1\right]^8$$

$$\Downarrow$$

$$\frac{dy}{dx} = 8\cdot\left[(x^2+1)^{10}+1\right]^7\cdot\left[10\cdot(x^2+1)^9\cdot 2x\right]$$

$$= 160x\cdot(x^2+1)^9\cdot\left[(x^2+1)^{10}+1\right]^7$$

$$y = \frac{\left[(x^3+7)^4+x\right]^5}{x^2+1}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{(x^2+1)\cdot 5\cdot\left[(x^3+7)^4+x\right]^4\cdot\left[4(x^3+7)^3\cdot 3x^2+1\right] - \left[(x^3+7)^4+x\right]^5\cdot 2x}{(x^2+1)^2}$$

$$= \frac{5(x^2+1)\cdot\left[(x^3+7)^4+x\right]^4\cdot\left[12x^2\cdot(x^3+7)^3+1\right] - 2x\cdot\left[(x^3+7)^4+x\right]^5}{(x^2+1)^2}$$

$$= \frac{\left[(x^3+7)^4+x\right]^4\cdot\left[5\cdot(x^2+1)\cdot\left[12x^2\cdot(x^3+7)^3+1\right] - 2x\cdot\left[(x^3+7)^4+x\right]\right]}{(x^2+1)^2}$$

$$x^3+y^3=1$$

$$\Downarrow$$

$$3x^2+\frac{dy^3}{dy}\frac{dy}{dx} = \frac{d(x^3)}{dx}+\frac{d(y^3)}{dx} = \frac{d(x^3+y^3)}{dx} = \frac{d(1)}{dx} = 0$$

1 = constant

$$= 3x^2+3y^2\frac{dy}{dx}$$

$$\Downarrow$$

$$3x^2+3y^2\frac{dy}{dx} = 0$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

i.e.,

$$x^3+y^3=1$$

$$\Downarrow$$

$$3x^2+3y^2\cdot y' = (x^3+y^3)' = 1' = 0$$

$$\Downarrow$$

$$y' = -\frac{3x^2}{3y^2} = \frac{-x^2}{y^2}$$

i.e.,

$$x^3+y^3=1$$

$$\Downarrow$$

$$3x^2+3y^2\frac{dy}{dx} = \frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(1) = 0$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = -\frac{x^2}{y^2}$$

$\boxed{\text{OR}}$

$$x^3 + y^3 = 1$$
$$\Downarrow$$
$$y^3 = 1 - x^3$$
$$\Downarrow$$
$$y = y^{\frac{3}{3}} = \left(y^{1/3}\right)^3 = \left(1-x^3\right)^{1/3}$$
$$\Downarrow$$
$$\frac{dy}{dx} = \frac{1}{3}\left(1-x^3\right)^{-\frac{2}{3}} \cdot \left(-3x^2\right)$$

$$= \frac{-x^2}{\left(1-x^3\right)^{-2/3}} = \frac{-x^2}{\left[\left(1-x^3\right)^{1/3}\right]^2}$$

$$= \frac{-x^2}{y^2}, \text{ as before!}$$

$$x^3 = y^5$$
$$\Downarrow$$
$$y = y^{5/5} = \left(y^5\right)^{1/5} = \left(x^3\right)^{1/5} = x^{3/5}$$
$$\Downarrow$$
$$\frac{dy}{dx} = \frac{3}{5} \cdot x^{-2/5}$$

$\boxed{\text{OR}}$

$$\boxed{\begin{array}{c} y = x^{3/5} \\ \Uparrow \\ x^3 = y^5 \end{array}}$$

(26)

$\boxed{\text{By the Chain Rule}}$

$$\Downarrow$$

$$3x^2 = \frac{d}{dx}(x^3) = \frac{d}{dx}(y^5) = \frac{d(y^5)}{dy} \cdot \frac{dy}{dx} = 5y^4 \frac{dy}{dx}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{3x^2}{5y^4} = \frac{3x^2}{5(x^{3/5})^4} = \frac{3}{5}\frac{x^2}{x^{12/5}}$$

$$= \frac{3}{5} \cdot \frac{x^{2 \cdot \frac{5}{5}}}{x^{12/5}} \cdot \frac{x^{-12/5}}{x^{-12/5}}$$

$$= \frac{3}{5} \frac{x^{\frac{10}{5}} \cdot x^{-12/5}}{\cancel{x}^{\left(\frac{12}{5} - 12/5\right)}}$$

$$= \frac{3}{5} \frac{x^{\frac{10-12}{5}}}{x^0} = \frac{3}{5} \frac{x^{-2/5}}{1}$$

$$= \frac{3}{5} \cdot x^{-2/5}$$, as before!

Still,

$$x^3 = y^5$$

$$\Downarrow$$

$$3x^2 = 5y^4 \frac{dy}{dx}$$

$$\Downarrow$$

$\frac{dy}{dx} = \frac{3x^2}{5y^4}$ is preferable!