MATH 2107-601 **EXAM #2** NAME: KEY (Print)

THURSDAY

NOVEMBER 1, 2001 NAME: ____________ (Signature)

[7:30 p.m. - 8:45 p.m.] STUDENT I. D.: ____________

INSTRUCTIONS: There are 2 parts to this exam, the main part plus an optional **EXTRA CREDIT** part (on the last page). The main part is worth 100 points while the extra credit part is worth 10 points. But any score over 100 will be truncated to 100.

Clarity of exposition (including proper spelling and punctuation is an integral part of a correct solution to any problem. In particular, **DO NOT PUT EQUAL SIGNS BETWEEN THINGS THAT ARE NOT EQUAL.** But do put them where they belong.

It is necessary to show all your work.

GOOD LUCK!

PLEASE SIGN THE FOLLOWING STATEMENT:
On my honor, I declare that the work that follows is entirely my own. With regard to all the questions on this exam, I have neither given nor received help from anyone [including myself, say, via any type of cheat sheet or device (e.g., cell phone)]. Nor have I used a calculator of any sort (i.e., regular or programmable).

NAME: ____________ (Signature)

PLEASE DO NOT WRITE BELOW THIS LINE:

15	1. a. 10	5. a. 5	**EXTRA-CREDIT:**
	b. 5	b. 5	1. 5
10	2. a. 5	c. 5	2. 5
	b. 5	d. 5	**TOTALS:**
30	3. 15	e. 5	Column 3: 10
	4. a. 5	f. 5	Column 2: 45
	b. 5	g. 5	Column 1: 55
	c. 5	h. 5	**Grand Total:** 110 ↘ 100
		i. 5	**Grade:** A+
	Total: 55	Total: 45	

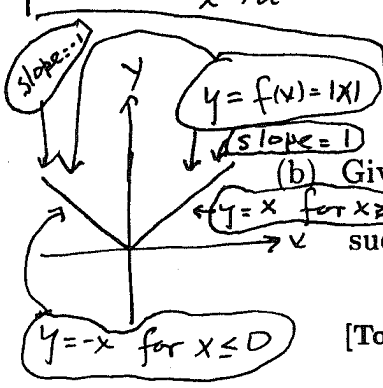
 NAME: KEY

1. (a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function and let $a\in\mathbb{R}$. Prove that

f is continuous at a whenever $f'(a)$ exists.

[10 points]
$$\lim_{h\to 0} f(a+h) = \lim_{h\to 0}\left\{ h\cdot\left[\frac{f(a+h)-f(a)}{h}\right] + f(a)\right\}$$
$$= \lim_{h\to 0} h \cdot \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right] + \lim_{h\to 0} f(a)$$
$$= 0 \cdot f'(a) + f(a)$$
$$= f(a)$$

$\boxed{\text{OR}}$
$$\lim_{x\to a} f(x) = \lim_{x\to a}\left\{(x-a)\cdot\left[\frac{f(x)-f(a)}{x-a}\right] + f(a)\right\}$$
$$= \lim_{x\to a}(x-a) \cdot \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right] + \lim_{x\to a} f(a)$$
$$= 0 \cdot f'(a) + f(a) = f(a)$$



(b) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ and a point $a\in\mathbb{R}$ such that f is continuous at a but $f'(a)$ does <u>not</u> exist.

[To get credit, you must give at least some indication of why f is continuous at a and why $f'(a)$ does not exist.]

[5 points] Let $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$, and let $a=0$.

Then, $f:\mathbb{R}\to\mathbb{R}$ is continuous at a because the graph of f is unbroken at $(a, f(a)) = (0, f(0)) = (0,0)$. [In fact, $f:\mathbb{R}\to\mathbb{R}$ is everywhere continuous since its graph is unbroken, i.e., is a single unbroken piece. In particular, f is continuous at $0\in\mathbb{R}$.]

On the other hand, $f'(0)$ does not exist because should $f'(0)$ exist, then $f'(0)$ would be the slope of the straight line tangent to the graph of f at $(0, f(0)) = (0,0)$. But, the graph of f has a corner at $(0, f(0)) = (0,0)$ & so can't have a tangent at that point as one can readily see from the graph of f. Alternatively, $f'_+(0) = 1$ while $f'_-(0) = -1$ so $f'(0)$ does not exist for otherwise $f'(0) = f'_+(0) = f'_-(0)$.

2. (a) State the **Chain Rule**. Be sure to state the hypotheses as well as the conclusion.

ANSWER:

The Chain Rule:

Hypotheses: $f'[g(x)]$ exists

[1 point]

and

$g'(x)$ exists

Conclusions: $(f \circ g)'(x)$ exists and

[4 points]

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x).$$

(b) Complete the following:

[5 points: 1 each]

$(F \cdot S)' = F \cdot S' + S \cdot F'$ $\boxed{\text{or}}$ $(F \cdot S)' = F' \cdot S + F \cdot S'$

$\left(\frac{T}{B}\right)' = \dfrac{B \cdot T' - T \cdot B'}{B^2}$

$(c \cdot F)' = c \cdot F'$

$(F + G)' = F' + G'$

$(F^n)' = n \cdot F^{n-1} \cdot F'$

3. Assume that the equation

$$x^2 + xy + +y^2 = 12$$

defines y as a differentiable function of x. Use implicit differentiation to find $\frac{dy}{dx}$. Then find the equation of the straight line tangent to the graph of $x^2 + xy + +y^2 = 12$ at the point $(2,2)$.

[15 points]

SOLUTION:

$$\Downarrow$$

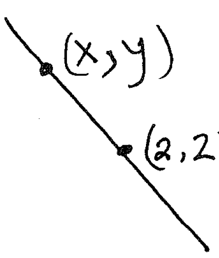
$$2x + x\cdot\frac{dy}{dx} + y\cdot 1 + 2y\cdot\frac{dy}{dx} = \frac{d\,(x^2+xy+y^2)}{dx} = \frac{d(12)}{dx} = 0$$

$$\|$$

$$2x + y + (x+2y)\cdot\frac{dy}{dx}$$

$$\Downarrow$$

$$m = \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=2}} = \left.-\frac{(2x+y)}{x+2y}\right|_{\substack{x=2\\y=2}} = -\frac{(2\cdot 2+2)}{2+2\cdot 2} = \frac{-6}{6} = -1$$



$$\frac{y-2}{x-2} = m = -1$$

$$\Downarrow$$

$$y-2 = -1\cdot(x-2) = -1\cdot x + 2$$

$$\Downarrow$$

$$y = -1\cdot x + 2 + 2 = -1\cdot x + 4$$

OR:

$$y = mx+b = -1\cdot x + b$$

$$2 = -1\cdot 2 + b = -2 + b$$

$$\Downarrow$$

$$b = 2+2 = 4$$

$$\Downarrow$$

$$y = -1\cdot x + b = -1\cdot x + 4 \Longrightarrow y = -1\cdot x + 4$$

4. For each of the following, use the various rules that we have developed [not the definition] to compute $f'(x)$ when f(x) is given.

Do **NOT** simplify your answers.

NOTE: DO NOT COMPUTE $f'(x)$ DIRECTLY FROM THE DEFINITION!

[15 points: 5 each]

a. $f(x) = \sqrt[6]{\frac{x^8+1}{x^4+1}} = \left(\frac{x^8+1}{x^4+1}\right)^{1/6}$

$$f'(x) = \frac{1}{6}\left[\frac{x^8+1}{x^4+1}\right]^{-5/6} \cdot \left[\frac{(x^4+1)\cdot 8x^7 - (x^8+1)\cdot 4x^3}{(x^4+1)^2}\right]$$

$$= \frac{1}{6}\left[\frac{x^8+1}{x^4+1}\right]^{-5/6} \cdot \left[\frac{4x^{11} + 8x^7 - 4x^3}{(x^4+1)^2}\right]$$

b. $f(x) = \frac{1 + \sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x$

$\Downarrow$

$$f'(x) = \sec x \cdot \tan x + \sec^2 x$$

$\boxed{\text{OR}}$

$$f'(x) = \frac{\cos x \cdot [0 + \cos x] - [1 + \sin x]\cdot(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x} = \frac{1 + \sin x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} + \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec^2 x + \sec x \cdot \tan x$$

c. $f(x) = (x^8 + x^4 + 2^8)^{100} \cdot (x^{1/7} - 2)$

$$f'(x) = (x^8 + x^4 + 2^8)^{100} \cdot \left[\frac{1}{7}x^{-6/7} + 0\right] + (x^{1/7} - 2) \cdot \left[100\,(x^8 + x^4 + 2^8)^{99} \cdot (8x^7 + 4x^3 + 0)\right]$$

5. For each of the following, use the various rules that we have developed [not the definition] to compute $f'(x)$ when f(x) is given.

Do **NOT** simplify your answers.

NOTE: DO NOT COMPUTE $f'(x)$ DIRECTLY FROM THE DEFINITION!

[45 points: 5 each]

a. $f(x) = \cos[\csc^6(4x) - 4] = \cos\left[\left(\csc(4x)\right)^6 - 4\right]$

$f'(x) = -\sin\left[\csc^6(4x)-4\right]\cdot\left[6\csc^5(4x)\right]\cdot\left[-\csc(4x)\cdot\cot(4x)\cdot 4\right]$

b. $f(x) = \sin(x^7) + \sin^7 x = \sin(x^7) + [\sin x]^7$

$f'(x) = \cos(x^7)\cdot 7x^6 + 7\cdot[\sin x]^6\cdot\cos x$

$= 7x^6\cdot\cos(x^7) + 7\cdot\sin^6 x\cdot\cos x$

c. $f(x) = \dfrac{x}{1-x} + \dfrac{1}{x-1} = \dfrac{x}{1-x} - \dfrac{1}{1-x} = \dfrac{x-1}{1-x} = -\dfrac{(1-x)}{1-x} = -1$

$\Downarrow$

$f'(x) = 0$

$\boxed{\text{OR}}$

$f'(x) = \dfrac{(1-x)\cdot 1 - x\cdot(0-1)}{(1-x)^2} + \dfrac{(x-1)\cdot 0 - 1\cdot(1-0)}{(x-1)^2}$

$= \dfrac{1-x+x-1}{(1-x)^2} = \dfrac{0}{(1-x)^2} = 0$

d. $f(x) = x^4 \cdot \tan\dfrac{1}{x^3} = x^4\cdot\tan(x^{-3})$

$f'(x) = x^4\cdot\left[\sec^2(x^{-3})\cdot(-3x^{-4})\right] + \tan(x^{-3})\cdot 4\cdot x^3$

$= -3\cdot\dfrac{x^4}{x^4}\cdot\sec^2\left(\dfrac{1}{x^3}\right) + 4x^3\cdot\tan\left(\dfrac{1}{x^3}\right)$

$= -3\cdot\sec^2\left(\dfrac{1}{x^3}\right) + 4x^3\cdot\tan\left(\dfrac{1}{x^3}\right).$

5. (Continued):

e. $f(x) = \sec x \cdot \sin x = \dfrac{1}{\cos x} \cdot \sin x = \tan x \Rightarrow f'(x) = \sec^2 x$

$\boxed{\text{OR}}$:

$$f'(x) = \sec x \cdot \cos x + \sin x \cdot \sec x \cdot \tan x$$

$$= \frac{1}{\cos x} \cdot \cos x + \sin x \cdot \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = 1 + \tan^2 x = \sec^2 x$$

f. $f(x) = \sec [\, \sin x \,]$

$$f'(x) = \sec[\sin x] \cdot \tan[\sin x] \cdot \cos x$$

g. $f(x) = \dfrac{8x^3 + 7}{4x^2 + 3}$

$$f'(x) = \frac{(4x^2+3) \cdot 24x^2 - [8x^3+7] \cdot 8x}{(4x^2+3)^2}$$

$$= \frac{32x^4 + 72x^2 - 56x}{(4x^2+3)^2}.$$

h. $f(x) = \sqrt[2]{\cot(4x) \cdot \tan(4x)} = \sqrt{\dfrac{1}{\tan(4x)} \cdot \tan(4x)} = \sqrt{1} = 1$

$\Downarrow$

$f'(x) = 0$

$\boxed{\text{OR}}$

$$f'(x) = \frac{1}{2}\left[\cot(4x) \cdot \tan(4x)\right]^{-1/2} \cdot \left[\cot(4x) \cdot \sec^2(4x) \cdot 4 + \tan(4x) \cdot [-\csc^2(4x) \cdot 4]\right]$$

$$= \frac{4}{2 \cdot \sqrt{\cot(4x) \cdot \tan(4x)}} \cdot \underbrace{\left[\frac{\cancel{\cos(4x)}}{\sin(4x)} \cdot \frac{1}{\cancel{\cos(4x)}} \cdot \frac{1}{\cos(4x)} - \frac{\cancel{\sin(4x)}}{\cos(4x)} \cdot \frac{1}{\cancel{\sin(4x)}} \cdot \frac{1}{\sin(4x)}\right]}_{0}$$

$$= \frac{2}{1} \cdot 0 = 0$$

i. $f(x) = [\cot(7x^6)]^5$

$$f'(x) = 5[\cot(7x^6)]^4 \cdot [-\csc^2(7x^6) \cdot 42x^5]$$

Note: $f(x) = [\cot(u)]^5$

$\Downarrow$

$$f'(x) = 5[\cot(u)]^4 \cdot [-\csc^2(u)] \cdot \frac{du}{dx}$$

EXTRA-CREDIT: [10 points: But, any score over 100 will be truncated to 100.]

1. *If* $y = \cos[\cos(\cos x)]$, then

$$\frac{dy}{dx} = -\sin\left[\cos(\cos x)\right]\cdot\left[-\sin(\cos x)\right]\cdot\left[-\sin x\right]$$
$$= -\sin\left[\cos(\cos x)\right]\cdot\left[\sin(\cos x)\right]\cdot\sin x$$

[5 points]

SOLUTION:

$\boxed{\text{DR}}$

$$Y = \cos\left[\cos(\cos x)\right] = \cos u, \text{ where } u = \cos(\cos x) = \cos v, \text{ where } v = \cos x.$$

Hence,

$$\frac{dy}{dx} = \frac{d\cos u}{du}\cdot\frac{du}{dx} = -\sin u\cdot\frac{du}{dx} = -\sin u\cdot\frac{d\cos v}{dx}$$
$$= -\sin u\cdot\frac{d\cos v}{dv}\cdot\frac{dv}{dx} = (-\sin u)\cdot(-\sin v)\cdot\frac{dv}{dx}$$
$$= (-\sin u)\cdot(-\sin v)\cdot\frac{d\cos x}{dx} = (\sin u)\cdot(\sin v)\cdot(-\sin x)$$
$$= -[\sin u]\cdot[\sin v]\cdot[\sin x] = -\sin\left[\cos(\cos x)\right]\cdot\left[\sin(\cos x)\right]\cdot\sin x$$

2. Let y be a differentiable function of x satisfying

$$\sin(x - 4y) = x^2\cdot\sin y\ .$$

Then

$$\frac{dy}{dx} = \frac{\cos(x-4y) - 2x\cdot\sin y}{x^2\cdot\cos y + 4\cdot\cos(x-4y)}$$

[5 points]

SOLUTION:

$$\sin(x-4y) = x^2\cdot\sin y$$
$$\Downarrow$$

$u = x-4y$

$$\cos u\cdot\frac{du}{dx} = \frac{d\sin u}{dx} = \frac{d\left[\sin(x-4y)\right]}{dx} = \frac{d\left[x^2\cdot\sin y\right]}{dx}$$
$$= x^2\cdot\frac{d(\sin y)}{dx} + \sin y\cdot 2x$$
$$= x^2\cdot\cos y\cdot\frac{dy}{dx} + 2x\cdot\sin y$$

$$\cos u\cdot\frac{du}{dx} = \cos(x-4y)\cdot\frac{d(x-4y)}{dx} = \cos(x-4y)\cdot\left[1 - 4\frac{dy}{dx}\right] = \cos(x-4y)\cdot 1 - 4\cos(x-4y)\cdot\frac{dy}{dx}$$

Hence, $\cos(x-4y) - 2x\cdot\sin y = \left[x^2\cdot\cos y + 4\cdot\cos(x-4y)\right]\cdot\frac{dy}{dx}$,

whence $$\frac{dy}{dx} = \frac{\cos(x-4y) - 2x\cdot\sin y}{x^2\cdot\cos y + 4\cdot\cos(x-4y)}\ .$$

Note: $\boxed{\text{for E-C-\#1}}$ $\boxed{\text{KEY}}$ $\boxed{\text{to page 7}}$

$$(g\circ f\circ h)'(x) := [g\circ(f\circ h)]'(x)$$
$$= g'[f\circ h(x)]\cdot(f\circ h)'(x)$$
$$= g'[f\circ h(x)]\cdot f'[h(x)]\cdot h'(x)$$

$\boxed{\text{OR}}$

$$(g\circ f\circ h)'(x) = [(g\circ f)\circ h]'(x)$$
$$= (g\circ f)'[h(x)]\cdot h'(x)$$
$$= g'[f(h(x))]\cdot f'[h(x)]\cdot h'(x)$$
$$= g'[(f\circ h)(x)]\cdot f'[h(x)]\cdot h'(x).$$

Here $g(x) = f(x) = h(x) = \cos x$.

Further $g'(x) = f'(x) = h'(x) = -\sin x$

So

$$\frac{dy}{dx} = \left(\cos[\cos(\cos x)]\right)' = (g\circ f\circ h)'(x)$$
$$= g'[f\circ h(x)]\cdot f'[h(x)]\cdot h'(x)$$
$$= -\sin[\cos(\cos x)]\cdot[-\sin(\cos x)]\cdot[-\sin x]$$
$$= -\sin[\cos(\cos x)]\cdot[\sin(\cos x)]\cdot\sin x$$