

$$f(x) = e^{4x^2+3x+1}$$

$$\Downarrow$$

$$f'(x) = e^{4x^2+3x+1} \cdot [8x+3]$$

$$f(x) = \cos(e^{2x})$$

$$\Downarrow$$

$$\begin{aligned} f'(x) &= -\sin(e^{2x}) \cdot e^{2x} \cdot 2 \\ &= -2 \cdot e^{2x} \cdot \sin(e^{2x}) \end{aligned}$$

$$f(x) = [\cos(e^{2x})]^4$$

$$\Downarrow$$

$$\begin{aligned} f'(x) &= 4 [\cos(e^{2x})]^3 \cdot [-\sin(e^{2x}) \cdot e^{2x} \cdot 2] \\ &= -8 \cdot e^{2x} \cdot \sin(e^{2x}) \cdot \cos^3(e^{2x}) \end{aligned}$$

$$f(x) = \sin x \cdot \cos x$$

$$\Downarrow$$

$$\begin{aligned} f'(x) &= \sin x \cdot [-\sin x] + \cos x \cdot [\cos x] \\ &= \cos^2 x - \sin^2 x = \cos(2x) \end{aligned}$$

$$\boxed{DR}$$

$$f(x) = \sin x \cdot \cos x = \frac{1}{2} \cdot 2 \sin x \cdot \cos x = \frac{1}{2} \sin(2x)$$

$$\Downarrow$$

$$f'(x) = \frac{1}{2} \cdot \cos(2x) \cdot 2 = \cos(2x) = \cos^2 x - \sin^2 x$$

(2)

Note: Recall that

$$\sin(A+B) = \sin(A) \cdot \cos(B) + \cos(A) \cdot \sin(B)$$

So, upon taking $A = B = x$, we have that

$$\begin{aligned} \sin(2x) &= \sin(x+x) = \sin x \cdot \cos x + \cos x \cdot \sin x \\ &= 2 \sin x \cdot \cos x \end{aligned}$$

Also,

$$\cos(A+B) = \cos(A) \cdot \cos(B) - \sin(A) \cdot \sin(B)$$

So, upon taking $A = B = x$, we have that

$$\begin{aligned} \cos(2x) &= \cos(x+x) = \cos x \cdot \cos x - \sin x \cdot \sin x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

Further, since $\sin^2 x + \cos^2 x = 1$

we have that

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 1 - \sin^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

and also that

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= \cos^2 x - (1 - \cos^2 x) \\ &= 2\cos^2 x - 1 \end{aligned}$$

which leads to the $\frac{1}{2}$ -angle formulas

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \& \quad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

i.e., $\cos^2\left(\frac{x}{2}\right) = \frac{1 + \cos(x)}{2} \quad \& \quad \sin^2\left(\frac{x}{2}\right) = \frac{1 - \cos(x)}{2}$.

Note: Be sure not to confuse ③

$$f(x) = \sin x \cdot \cos x = \cos x \cdot \sin x$$

with either

$$f(x) = \sin[\cos x] \quad \text{or} \quad f(x) = \cos[\sin x]$$

Indeed

$$f(x) = \sin x \cdot \cos x$$

↓

$$\begin{aligned} f'(x) &= \sin x \cdot (\cos x)' + \cos x \cdot (\sin x)' \\ &= \sin x \cdot (-\sin x) + \cos x \cdot \cos x \\ &= \cos^2 x - \sin^2 x, \end{aligned}$$

as we saw above, whereas

$$f(x) = \sin[\cos x]$$

↓

$$\begin{aligned} f'(x) &= \cos[\cos x] \cdot (\cos x)' \\ &= \cos[\cos x] \cdot (-\sin x) \\ &= -\sin x \cdot \cos[\cos x] \end{aligned}$$

while

$$f(x) = \cos[\sin x]$$

↓

$$\begin{aligned} f'(x) &= -\sin[\sin x] \cdot (\sin x)' \\ &= -\sin[\sin x] \cdot \cos x \\ &= -\cos x \cdot \sin[\sin x]. \end{aligned}$$

Indeed, one has

$$\begin{aligned}(f \cdot g)'(x) &= (f(x) \cdot g(x))' \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x)\end{aligned}$$

versus

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

Recall: $(f \cdot g)(x) := f(x) \cdot g(x)$
 = the pointwise product
 of $f(x)$ & $g(x)$,

whereas

$$f \circ g(x) := f[g(x)]$$

i.e., $f \circ g$ = the composite of the
inner function g followed
 by the outer function
 f

i.e., $f \cdot g$ = the pointwise product
 of f & g

while $f \circ g$ = the composition
 product of f & g

(5)

Junk of x

$$f(x) = \sin [J(x)]$$

$$\Downarrow$$

$$f'(x) = \cos [J(x)] \cdot J'(x)$$

$$f(x) = \cos [J(x)]$$

$$\Downarrow$$

$$f'(x) = -\sin [J(x)] \cdot J'(x)$$

$$\boxed{DR}$$

$$y = f(x) = \sin(u)$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = \cos(u) \cdot \frac{du}{dx}$$

$$y = f(x) = \cos(u)$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = -\sin(u) \frac{du}{dx}$$

$$\boxed{DR}$$

$$y = \sin(u)$$

$$\Downarrow$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d \sin(u)}{dx} \\ &= \frac{d \sin(u)}{du} \cdot \frac{du}{dx} \\ &= \cos(u) \cdot \frac{du}{dx} \end{aligned}$$

$$y = \cos(u)$$

$$\Downarrow$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d \cos(u)}{dx} \\ &= \frac{d \cos(u)}{du} \cdot \frac{du}{dx} \\ &= -\sin(u) \frac{du}{dx} \end{aligned}$$

(6)

$$y = f(x) = \tan[J(x)]$$

↓

$$\frac{dy}{dx} = f'(x) = \sec^2[J(x)] \cdot J'(x)$$

OR

$$y = \tan(u)$$

↓

$$\frac{dy}{dx} = \frac{d \tan(u)}{dx} = \frac{d \tan(u)}{du} \cdot \frac{du}{dx}$$

$$= \sec^2(u) \cdot \frac{du}{dx}$$

$$y = f(x) = \cot[J(x)]$$

↓

$$\frac{dy}{dx} = f'(x) = -\csc^2[J(x)] \cdot J'(x)$$

OR

$$y = \cot(u)$$

↓

$$\frac{dy}{dx} = \frac{d \cot(u)}{dx} = \frac{d \cot(u)}{du} \cdot \frac{du}{dx}$$

$$= -\csc^2(u) \cdot \frac{du}{dx}$$

(7)

$$y = f(x) = \sec[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = \sec[J(x)] \cdot \tan[J(x)] \cdot J'(x)$$

[OR]

$$y = \sec(u)$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d \sec(u)}{dx} = \frac{d \sec(u)}{du} \cdot \frac{du}{dx}$$

$$= \sec(u) \cdot \tan(u) \cdot \frac{du}{dx}$$

$$y = f(x) = \csc[J(x)]$$

$$\Downarrow$$

$$\frac{dy}{dx} = f'(x) = -\csc[J(x)] \cdot \cot[J(x)] \cdot J'(x)$$

[OR]

$$y = \csc(u)$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{d \csc(u)}{dx} = \frac{d \csc(u)}{du} \cdot \frac{du}{dx}$$

$$= -\csc(u) \cdot \cot(u) \cdot \frac{du}{dx}$$

8

$$y = f(x) = e^{J(x)}$$

⇓

$$\frac{dy}{dx} = f'(x) = e^{J(x)} \cdot J'(x)$$

[OR]

$$y = e^u$$

⇓

$$\frac{dy}{dx} = \frac{de^u}{dx} = \frac{de^u}{du} \cdot \frac{du}{dx}$$

$$= e^u \cdot \frac{du}{dx}$$

$$y = \ln[J(x)] = \log_e [J(x)], \text{ for } J(x) > 0$$

⇓

$$\frac{dy}{dx} = f'(x) = \frac{1}{J(x)} \cdot J'(x) = \frac{J'(x)}{J(x)}$$

[OR]

$$y = \ln(u) \text{ for } u > 0$$

⇓

$$\frac{dy}{dx} = \frac{d \ln(u)}{dx} = \frac{d \ln(u)}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot \frac{du}{dx}$$

(9)

$$y = f(x) = [J(x)]^c, \text{ where } c = \text{a constant}$$

⇓

$$\frac{dy}{dx} = f'(x) = c \cdot [J(x)]^{c-1} \cdot J'(x)$$

OR

a constant

↙

$$y = u$$

↑

a variable = a function of x

⇓

$$\begin{aligned} \frac{dy}{dx} &= \frac{d u^c}{dx} = \frac{d u^c}{du} \cdot \frac{du}{dx} \\ &= c \cdot u^{c-1} \cdot \frac{du}{dx} \end{aligned}$$

(10)

$$f(x) = \sin[\cos(x^2)] = \sin[J(x)]$$

$$\Downarrow$$

$$f'(x) = \cos[J(x)] \cdot J'(x)$$

OR

$$f(x) = \sin(u)$$

$$\Downarrow$$

$$f'(x) = \cos(u) \cdot \frac{du}{dx}$$

So:

$$f(x) = \sin[\cos(x^2)]$$

$$\Downarrow$$

$$f'(x) = \cos[\cos(x^2)] \cdot [-\sin(x^2) \cdot (2x)]$$

OR

$$y = \sin[\cos(x^2)]$$

$$\frac{dy}{dx} = \frac{d \sin[\cos(x^2)]}{dx}$$

$$= \frac{d \sin[\cos(x^2)]}{d \cos(x^2)} \cdot \frac{d \cos(x^2)}{dx}$$

$$= \cos[\cos(x^2)] \cdot \frac{d \cos(x^2)}{dx^2} \cdot \frac{dx^2}{dx}$$

$$= \cos[\cos(x^2)] \cdot [-\sin(x^2) \cdot 2x]$$

$$f(x) = \left(\sin [\cos(x^2)] \right)^3 = (J(x))^3 \quad (11)$$

↓

$$f'(x) = 3 \cdot (J(x))^2 \cdot J'(x)$$

$$= 3 \cdot \left(\sin [\cos(x^2)] \right)^2 \cdot \left[\cos [\cos(x^2)] \cdot (-\sin(x^2) \cdot 2x) \right]$$

[OR]

$$y = f(x) = u^3 \quad \text{where } u = \sin[\cos(x^2)]$$

$$= \sin(v) \quad \text{where}$$

$$v = \cos(x^2)$$

$$= \cos(w)$$

$$\text{where } w = x^2$$

↓

$$f'(x) = \frac{dy}{dx} = \frac{d u^3}{dx} = \frac{d u^3}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \frac{du}{dx}$$

$$= 3u^2 \cdot \frac{d \sin(v)}{dv} = 3u^2 \cdot \frac{d \sin(v)}{dv} \cdot \frac{dv}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot \frac{dv}{dx} = 3u^2 \cdot \cos(v) \cdot \frac{d \cos(w)}{dw}$$

$$= 3u^2 \cdot \cos(v) \cdot \frac{d \cos(w)}{dw} \cdot \frac{dw}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot (-\sin(w)) \cdot \frac{dw}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot (-\sin(w)) \cdot \frac{d x^2}{dx}$$

$$= 3u^2 \cdot \cos(v) \cdot (-\sin(w)) \cdot 2x$$

$$= 3 \left(\sin [\cos(x^2)] \right)^2 \cdot \cos [\cos(x^2)] \cdot (-\sin(x^2)) \cdot 2x$$

$$= -6x \cdot \left(\sin [\cos(x^2)] \right)^2 \cdot \cos [\cos(x^2)] \cdot \sin(x^2)$$

$$f(x) = x^7 + x^5 \Rightarrow f'(x) = 7x^6 + 5x^4 \quad (12)$$

$$f(x) = -6x^3 + 12x^2 - 4x + 7$$

$$\Downarrow$$

$$f'(x) = -18x^2 + 24x - 4$$

$$f(x) = \frac{1}{10} \cdot x^{10} + \frac{1}{9} \cdot x^9 + \frac{1}{8} \cdot x^8$$

$$\Downarrow$$

$$f'(x) = x^9 + x^8 + x^7$$

$$f(x) = (3x+5)^2 = 9x^2 + 30x + 25$$

$$\Downarrow$$

$$f'(x) = 18x + 30$$

$$\boxed{\text{OR}} \quad f(x) = (3x+5)^2 \Rightarrow f'(x) = 2 \cdot (3x+5)^1 \cdot 3$$

$$= 6 \cdot (3x+5) = 18x + 30$$

$$f(x) = (-2x^2+1)^3 = (-2x^2+1) \cdot (-2x^2+1)^2$$

$$= (-2x^2+1) \cdot (4x^4 - 4x^2 + 1)$$

$$= -8x^6 + 8x^4 - 2x^2 + 4x^4 - 4x^2 + 1$$

$$= -8x^6 + 12x^4 - 6x^2 + 1$$

$$\Downarrow$$

$$f'(x) = -48x^5 + 48x^3 - 12x$$

$$\boxed{\text{OR}} \quad f(x) = (-2x^2+1)^3 = (a+b)^3 \text{ where } a = -2x^2 \text{ \& } b = 1$$

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$$= a^3 + 3a^2 + 3a + 1$$

$$= (-2x^2)^3 + 3 \cdot (-2x^2)^2 + 3 \cdot (-2x^2) + 1$$

$$= -8x^6 + 3 \cdot 4x^4 - 6x^2 = -8x^6 + 12x^4 - 6x^2 + 1$$

$$\Downarrow$$

$$f'(x) = -48x^5 + 48x^3 - 12x$$

$$\boxed{\text{OR}} \quad f(x) = (-2x^2+1)^3 \Rightarrow f'(x) = 3 \cdot (-2x^2+1)^2 \cdot (-4x)$$

$$= -12x \cdot (4x^4 - 4x^2 + 1)$$

$$= -48x^5 + 48x^3 - 12x$$

$$\begin{aligned}
 f(x) &= (3x^2 - 7x + 1)(x^2 + x - 1) \\
 &= 3x^2 \cdot (x^2 + x - 1) - 7x \cdot (x^2 + x - 1) + 1 \cdot (x^2 + x - 1) \\
 &= 3x^4 + \underbrace{3x^3 - 3x^2 - 7x^3 - 7x^2 + 7x + x^2 + x - 1} \\
 &= 3x^4 - 4x^3 - 9x^2 + 8x - 1
 \end{aligned}$$

[OR]

$$\begin{array}{r}
 3x^2 - 7x + 1 \\
 \hline
 x^2 + x - 1 \\
 \hline
 3x^4 - 7x^3 + x^2 \\
 3x^3 - 7x^2 + x \\
 - 3x^2 + 7x - 1 \\
 \hline
 \end{array}$$

so $f(x) = 3x^4 - 4x^3 - 9x^2 + 8x - 1$

∴

$$f'(x) = 12x^3 - 12x^2 - 18x + 8$$

[OR]

$$f(x) = (3x^2 - 7x + 1) \cdot (x^2 + x - 1)$$

∴

$$f'(x) = (3x^2 - 7x + 1) \cdot (2x + 1) + (x^2 + x - 1) \cdot (6x - 7)$$

$$\begin{aligned}
 &= 6x^3 + 3x^2 - 14x^2 - 7x + 2x + 1 \\
 &\quad + 6x^3 - 7x^2 + 6x^2 - 7x - 6x + 7 \\
 &= 12x^3 - 4x^2 - 8x^2 - 14x - 4x + 8 \\
 &= 12x^3 - 12x^2 - 18x + 8
 \end{aligned}$$

$$f(x) = \frac{5x+6}{x^2+1}$$

↓

$$f'(x) = \frac{(x^2+1) \cdot 5 - (5x+6) \cdot 2x}{(x^2+1)^2}$$

$$= \frac{5x^2+5-10x^2-12x}{(x^2+1)^2} = \frac{-5x^2-12x+5}{(x^2+1)^2}$$

$$f(x) = \frac{x^3-x}{x^5-x^3+1}$$

↓

$$f'(x) = \frac{(x^5-x^3+1) \cdot (3x^2-1) - (x^3-x) \cdot (5x^4-3x^2)}{(x^5-x^3+1)^2}$$

$$= \frac{3x^7-x^5-3x^5+x^3+3x^2-1 - (5x^7-3x^5-5x^5+3x^3)}{(x^5-x^3+1)^2}$$

$$= \frac{(3-5)x^7 + (3+5-1-3)x^5 + (1-3)x^3 + 3x^2-1}{(x^5-x^3+1)^2}$$

$$= \frac{-2x^7 + 4x^5 - 2x^3 + 3x^2 - 1}{(x^5-x^3+1)^2}$$

$$f(x) = \frac{x^2}{2} = \frac{1}{2} \cdot x^2$$

$$\Downarrow$$

$$f'(x) = \frac{1}{2} \cdot 2 \cdot x^1 = x$$

OR

$$f(x) = \frac{x^2}{2}$$

$$\Downarrow$$

$$f'(x) = \frac{2 \cdot (2x) - x^2 \cdot 0}{2^2} = \frac{4x}{4} = x$$

$$f(x) = (x^3+1)(x^2+1)(x+1)$$

$$= (x^3+1)(x^3+x^2+x+1)$$

$$= x^6 + x^5 + x^4 + \overbrace{x^3} + x^3 + x^2 + x + 1$$

$$= x^6 + x^5 + x^4 + 2x^3 + x^2 + x + 1$$

$$\Downarrow$$

$$f'(x) = 6x^5 + 5x^4 + 4x^3 + 6x^2 + 2x + 1$$

OR

$$f(x) = (x^3+1)(x^2+1)(x+1) = u \cdot v \cdot w$$

$$\Downarrow$$

$$f'(x) = u' \cdot v \cdot w + u \cdot v' \cdot w + u \cdot v \cdot w'$$

$$= 3x^2 \cdot (x^2+1) \cdot (x+1) + (x^3+1) \cdot 2x \cdot (x+1) + (x^3+1) \cdot (x^2+1) \cdot 1$$

$$= 3x^2 \cdot (x^3 + x^2 + x + 1) + (2x^4 + 2x)(x+1) + x^5 + x^3 + x^2 + 1$$

$$= 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + x^5 + x^3 + x^2 + 1$$

$$= 6x^5 + 5x^4 + 4x^3 + 6x^2 + 2x + 1$$

(17)

$$f(x) = \frac{(x^3 - 2x^2 + 7x + 1) \cdot (x^2 - x + 3)}{x^3 + 1}$$

Now,

$$x^3 - 2x^2 + 7x + 1$$

$$x^2 - x + 3$$

$$x^5 - 2x^4 + 7x^3 + x^2$$

$$- x^4 + 2x^3 - 7x^2 - x$$

$$3x^3 - 6x^2 + 21x + 3$$

$$x^5 - 3x^4 + 12x^3 - 12x^2 + 20x + 3$$

Hence,

$$f(x) = \frac{x^5 - 3x^4 + 12x^3 - 12x^2 + 20x + 3}{x^3 + 1}$$

$$\Downarrow$$

$$f'(x) = \frac{(x^3 + 1) \cdot (5x^4 - 12x^3 + 36x^2 - 24x + 20) - (x^5 - 3x^4 + 12x^3 - 12x^2 + 20x + 3) \cdot 3x^2}{(x^3 + 1)^2}$$

$$5x^7 - 12x^6 + 36x^5 - 24x^4 + 20x^3$$

$$+ 5x^4 - 12x^3 + 36x^2 - 24x + 20$$

$$- 3x^7 + 9x^6 - 36x^5 + 36x^4 - 60x^3 - 9x^2$$

=

$$(x^3 + 1)^2$$

$$= \frac{2x^7 - 3x^6 + 17x^4 - 52x^3 + 27x^2 - 24x + 20}{(x^3 + 1)^2}$$

$$(x^3 + 1)^2$$

$$f(x) = \frac{(x^2+1) \cdot (x^4+1)}{(x^3+1) \cdot (x^5+1)} = \frac{(x^4+1) \cdot (x^2+1)}{(x^5+1) \cdot (x^3+1)} = \frac{x^6 + x^4 + x^2 + 1}{x^8 + x^5 + x^3 + 1}$$

↓

$$f'(x) = \frac{(x^8 + x^5 + x^3 + 1) \cdot (6x^5 + 4x^3 + 2x) - (x^6 + x^4 + x^2 + 1) \cdot (8x^7 + 5x^4 + 3x^2)}{(x^8 + x^5 + x^3 + 1)^2}$$

Now,

$$\frac{6x^5 + 4x^3 + 2x}{x^8 + x^5 + x^3 + 1}$$

$$6x^{13} + 4x^{11} + 2x^9$$

$$6x^{10} + 4x^8 + 2x^6$$

$$6x^8 + 4x^6 + 2x^4$$

$$6x^5 + 4x^3 + 2x$$

$$6x^{13} + 4x^{11} + 6x^{10} + 2x^9 + 10x^8 + 6x^6 + 6x^5 + 2x^4 + 4x^3 + 2x$$

while

$$8x^7 + 5x^4 + 3x^2$$

$$x^6 + x^4 + x^2 + 1$$

$$8x^{13} + 5x^{10} + 3x^8$$

$$8x^{11} + 5x^8 + 3x^6$$

$$8x^9 + 5x^6 + 3x^4$$

$$8x^7 + 5x^4 + 3x^2$$

$$8x^{13} + 8x^{11} + 5x^{10} + 8x^9 + 8x^8 + 8x^7 + 8x^6 + 8x^4 + 3x^2$$

Hence,

$$f'(x) = \frac{(6-8)x^{13} + (4-8)x^{11} + (6-5)x^{10} + (2-8)x^9 + (10-8)x^8 - 8x^7 + (6-8)x^6 + 6x^5 + (2-8)x^4 + 4x^3 - 3x^2 + 2x}{(x^8 + x^5 + x^3 + 1)^2}$$

$$= \frac{-2x^{13} - 4x^{11} + x^{10} - 6x^9 + 2x^8 - 8x^7 - 2x^6 + 6x^5 - 6x^4 + 4x^3 - 3x^2 + 2x}{(x^8 + x^5 + x^3 + 1)^2}$$

OR

$$f(x) = \frac{(x^2+1) \cdot (x^4+1)}{(x^3+1) \cdot (x^5+1)}$$

(19)

↓

$$f'(x) = \frac{(x^3+1) \cdot (x^5+1) \cdot [(x^2+1) \cdot 4x^3 + (x^4+1) \cdot 2x] - (x^2+1) \cdot (x^4+1) \cdot [(x^3+1) \cdot 5x^4 + (x^5+1) \cdot 3x^2]}{(x^3+1)^2 \cdot (x^5+1)^2}$$

$$= \frac{(x^8+x^3+x^5+1) \cdot [4x^5+4x^3+2x^5+2x] - (x^6+x^2+x^4+1) \cdot [5x^7+5x^4+3x^7+3x^2]}{(x^3+1)^2 \cdot (x^5+1)^2}$$

$$= \frac{(x^8+x^5+x^3+1) \cdot (6x^5+4x^3+2x) - (x^6+x^4+x^2+1) \cdot (8x^7+5x^4+3x^2)}{(x^3+1)^2 \cdot (x^5+1)^2}$$

Now,

$$\frac{6x^5 + 4x^3 + 2x}{x^8 + x^5 + x^3 + 1}$$

$$6x^{13} + 4x^{11} + 2x^9$$

$$6x^{10} + 4x^8 + 2x^6$$

$$6x^8 + 4x^6 + 2x^4$$

$$6x^5 + 4x^3 + 2x$$

$$6x^{13} + 4x^{11} + 6x^{10} + 2x^9 + 10x^8 + 6x^6 + 6x^5 + 2x^4 + 4x^3 + 2x$$

while

$$8x^7 + 5x^4 + 3x^2$$

$$x^6 + x^4 + x^2 + 1$$

$$8x^{13} + 5x^{10} + 3x^8$$

$$8x^{11}$$

$$8x^9$$

$$8x^7$$

$$+ 5x^4 + 3x^2$$

$$8x^{13} + 8x^{11} + 5x^{10} + 8x^9 + 8x^8 + 8x^7 + 8x^6 + 8x^4 + 3x^2$$

Hence, as before,

$$f'(x) = \frac{-2x^{13} - 4x^{11} + x^{10} - 6x^9 + 2x^8 - 8x^7 - 2x^6 + 6x^5 - 6x^4 + 4x^3 - 3x^2 + 2x}{(x^3+1)^2 \cdot (x^5+1)^2}$$

$$f(x) = [g(x)]^2 = g(x) \cdot g(x)$$

$$\Downarrow$$

$$f'(x) = g(x) \cdot g'(x) + g(x) \cdot g'(x)$$

$$= 2 \cdot g(x) \cdot g'(x)$$

$$\Downarrow$$

$$f'(1) = 2 \cdot g(1) \cdot g'(1)$$

Note: $f(1) = [g(1)]^2$

But $f'(1) \neq [f(1)]' = 0$

Since $f(1) = [g(1)]^2 = \text{a constant!!!}$

In other words, to compute $f'(1)$ you must differentiate 1st & then plug-in, as opposed to plugging in (& getting a constant) & then differentiating.

I hope none of you thought that

$$f'(1) = [f(1)]'$$

which is false. For example,

If $g(x) = x^3 + 1$, then

$$f(x) = g(x) \cdot g(x) = (x^3 + 1)^2 = x^6 + 2x^3 + 1$$

$$\text{so } f'(x) = 6x^5 + 6x^2 \quad \boxed{\text{OR}} \quad f'(x) = 2 \cdot (x^3 + 1) \cdot 3x^2 = 6x^2 \cdot (x^3 + 1) = 6x^5 + 6x^2$$

$$\text{so } f'(1) = 6 \cdot 1^5 + 6 \cdot 1^2 = 6 \cdot 1 + 6 \cdot 1 = 6 + 6 = 12$$

$$\text{where as } f(1) = (1^3 + 1)^2 = (1 + 1)^2 = 2^2 = 4 \text{ so } (f(1))' = 4' = 0 \text{ !!!}$$

(21)

$$y = (x^2 + 2x + 3)^2$$

↓

$$\frac{dy}{dx} = 2 \cdot (x^2 + 2x + 3)^1 \cdot (2x + 2) = 2(x^2 + 2x + 3) \cdot 2 \cdot (x + 1) \\ = 4 \cdot (x + 1) \cdot (x^2 + 2x + 3)$$

OR

$$y = (x^2 + 2x + 3)^2 = ?$$

Now, $a + b + c$

$$\begin{array}{r}
 a + b + c \\
 \hline
 a^2 \qquad + ab \qquad + ac \\
 \quad b^2 \quad + \quad ab \qquad \cdot \qquad + bc \\
 \qquad \quad + c^2 \qquad \qquad \quad ac \quad + bc
 \end{array}$$

$$a^2 + b^2 + c^2 + 2(ab + ac + bc)$$

↷
abc
↶

Hence $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$

So

$$\begin{aligned}
 y &= (x^2 + 2x + 3)^2 = x^4 + 4x^2 + 9 + 2 \cdot (2x^3 + 3x^2 + 6x) \\
 &= x^4 + 4x^2 + 9 + 4x^3 + 6x^2 + 12x \\
 &= x^4 + 4x^3 + 10x^2 + 12x + 9
 \end{aligned}$$

Note:

$$\begin{array}{r}
 x^2 + 2x + 3 \\
 x^2 + 2x + 3 \\
 \hline
 x^4 + 2x^3 + 3x^2 \\
 \quad 2x^3 + 4x^2 + 6x \\
 \qquad \quad 3x^2 + 6x + 9 \\
 \hline
 x^4 + 4x^3 + 10x^2 + 12x + 9
 \end{array}$$

Hence

$$y = (x^2 + 2x + 3)^2 = x^4 + 4x^3 + 10x^2 + 12x + 9$$

↓

$$\frac{dy}{dx} = 4x^3 + 12x^2 + 20x + 12$$

The question is, therefore,

$$\text{Does } 4x^3 + 12x^2 + 20x + 12 \stackrel{?}{=} 4(x+1) \cdot (x^2 + 2x + 3)?$$

If so, -1 is a root of $4x^3 + 12x^2 + 20x + 12$

$$\text{So } \begin{array}{r} -1 \overline{) 4 \quad 12 \quad 20 \quad 12} \\ \underline{-4 \quad -8 \quad -12} \\ 4 \quad 8 \quad 12 \quad \underline{0} \end{array} \leftarrow \text{BINGO!}$$

↓

$$4x^2 + 8x + 12 \cdot x^0$$

$$\begin{aligned} \text{So } 4x^3 + 12x^2 + 20x + 12 &= [x - (-1)] \cdot [4x^2 + 8x + 12] \\ &= (x+1) \cdot 4 \cdot [x^2 + 2x + 3] \\ &= 4 \cdot (x+1) \cdot (x^2 + 2x + 3) \end{aligned}$$

Note: One also has that

$$\begin{array}{r} x^2 + 2x + 3 \\ 4x + 4 \\ \hline 4x^3 + 8x^2 + 12x \\ \quad 4x^2 + 8x + 12 \\ \hline 4x^3 + 12x^2 + 20x + 12 \end{array} \quad !!!$$

$$y = \left(\frac{x^3}{3} + 1\right)^5 + \left(\frac{x^2}{2} + 1\right)^4 + 3$$

$$\Downarrow$$

$$\frac{dy}{dx} = 5 \cdot \left(\frac{x^3}{3} + 1\right)^4 \cdot (x^2) + 4 \cdot \left(\frac{x^2}{2} + 1\right)^3 \cdot x$$

$$= 5x^2 \cdot \left(\frac{x^3}{3} + 1\right)^4 + 4x \cdot \left(\frac{x^2}{2} + 1\right)^3$$

$$y = \left[(x^2+1)^{10} + 1\right]^8$$

$$\Downarrow$$

$$\frac{dy}{dx} = 8 \cdot \left[(x^2+1)^{10} + 1\right]^7 \cdot \left[10 \cdot (x^2+1)^9 \cdot 2x\right]$$

$$= 160x \cdot (x^2+1)^9 \cdot \left[(x^2+1)^{10} + 1\right]^7$$

$$y = \frac{\left[(x^3+7)^4 + x\right]^5}{x^2+1}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{(x^2+1) \cdot 5 \cdot \left[(x^3+7)^4 + x\right]^4 \cdot \left[4(x^3+7)^3 \cdot 3x^2 + 1\right] - \left[(x^3+7)^4 + x\right]^5 \cdot 2x}{(x^2+1)^2}$$

$$= \frac{5(x^2+1) \cdot \left[(x^3+7)^4 + x\right]^4 \cdot \left[12x^2 \cdot (x^3+7)^3 + 1\right] - 2x \cdot \left[(x^3+7)^4 + x\right]^5}{(x^2+1)^2}$$

$$= \frac{\left[(x^3+7)^4 + x\right]^4 \cdot \left[5 \cdot (x^2+1) \cdot \left[12x^2 \cdot (x^3+7)^3 + 1\right] - 2x \cdot \left[(x^3+7)^4 + x\right]\right]}{(x^2+1)^2}$$

(24)

$$x^3 + y^3 = 1$$

⇓

$$3x^2 + \frac{dy^3}{dy} \frac{dy}{dx} = \frac{d(x^3 + y^3)}{dx} = \frac{d(1)}{dx} = 0$$

↑
1 = constant

||

$$3x^2 + 3y^2 \frac{dy}{dx}$$

⇓

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

⇓

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

i.e.,

$$x^3 + y^3 = 1$$

⇓

$$3x^2 + 3y^2 \cdot y' = (x^3 + y^3)' = 1' = 0$$

⇓

$$y' = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

i.e.,

$$x^3 + y^3 = 1$$

⇓

$$3x^2 + 3y^2 \frac{dy}{dx} = \frac{d(x^3 + y^3)}{dx} = \frac{d(1)}{dx} = 0$$

⇓

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

OR

$$x^3 + y^3 = 1$$

$$\Downarrow$$

$$y^3 = 1 - x^3$$

$$\Downarrow$$

$$y = y^{\frac{3}{3}} = (y^{\frac{1}{3}})^3 = (1 - x^3)^{\frac{1}{3}}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{1}{3} (1 - x^3)^{-\frac{2}{3}} \cdot (-3x^2)$$

$$= \frac{-x^2}{(1 - x^3)^{-2/3}} = \frac{-x^2}{[(1 - x^3)^{1/3}]^2}$$

$$= -\frac{x^2}{y^2}, \text{ as before!}$$

$$x^3 = y^5$$

$$\Downarrow$$

$$y = y^{5/5} = (y^5)^{1/5} = (x^3)^{1/5} = x^{3/5}$$

$$\Downarrow$$

$$\frac{dy}{dx} = \frac{3}{5} \cdot x^{-2/5}$$

OR

$$y = x^{3/5}$$

$$\uparrow$$

$$x^3 = y^5$$

By the Chain Rule

$$3x^2 = \frac{d}{dx}(x^3) = \frac{d}{dx}(y^5) = \frac{d}{dy}(y^5) \cdot \frac{dy}{dx} = 5y^4 \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2}{5y^4} = \frac{3x^2}{5(x^{3/5})^4} = \frac{3x^2}{5x^{12/5}}$$

$$= \frac{3}{5} \cdot \frac{x^{2 \cdot \frac{5}{5}}}{x^{12/5}} \cdot \frac{x^{-12/5}}{x^{-12/5}}$$

$$= \frac{3}{5} \frac{x^{\frac{10}{5}} \cdot x^{-12/5}}{x^{(12/5 - 12/5)}}$$

$$= \frac{3}{5} \frac{x^{\frac{10-12}{5}}}{x^0} = \frac{3}{5} \frac{x^{-2/5}}{1}$$

$$= \frac{3}{5} \cdot x^{-2/5}, \text{ as before!}$$

Still,

$$x^3 = y^5$$

↓

$$3x^2 = 5y^4 \frac{dy}{dx}$$

↓

$$\frac{dy}{dx} = \frac{3x^2}{5y^4} \text{ is preferable!}$$