Math 2107

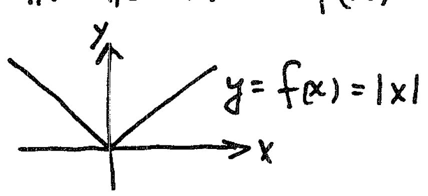
A function $f : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable provided its graph

1) is unbroken
2) has no corners
3) has no vertical tangents
4) has no points at which it oscillates wildly.

Comments: ① If $f'(a)$ exists, then f is continuous at a, i.e., $\lim_{h\to 0} f(a+h) = f(a)$. Since the graph of an everywhere continuous function $f : \mathbb{R} \to \mathbb{R}$ is unbroken, this explains (1).

② Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

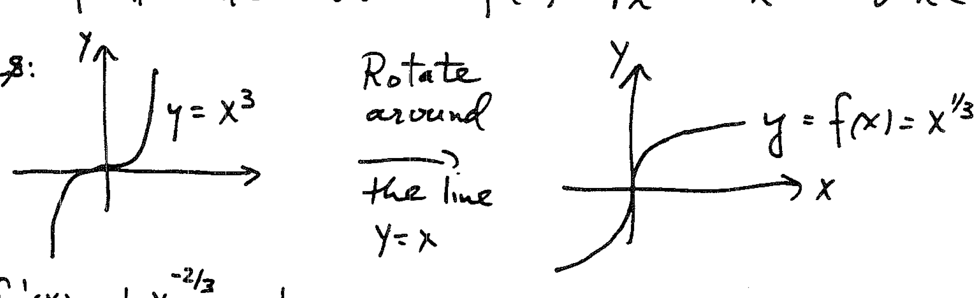
Picture:



Then $f : \mathbb{R} \to \mathbb{R}$ is everywhere continuous since its graph is unbroken; but $f'(0)$ does not exist since the graph of f has a corner at $(0, f(0)) = (0,0)$ & hence the graph of f does not have a tangent at $(0, f(0))$. But, by definition, $f'(0) =$ the slope of the straight line tangent to the graph of f at $(0, f(0))$.

③ Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \sqrt[3]{x} = x^{1/3}$ $\forall x \in \mathbb{R}$.

Pictures:



Then $f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$ if $x \neq 0$ while

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = +\infty$$

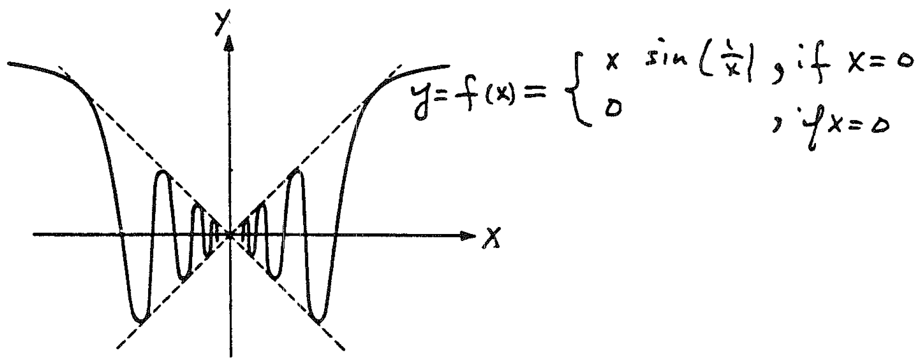
Thus $f'(0)$ does not exist.

Geometrically, we see that the graph of $f: \mathbb{R} \to \mathbb{R}$ (where $f(x) = x^{1/3}$) has a vertical tangent at $(0, f(0)) = (0,0)$.

Except at $(0, f(0))$ the tangent to the graph of f at $(x, f(x))$ always has a positive slope $= f'(x) = \frac{1}{3}\frac{1}{x^{2/3}} = \frac{1}{3}\left(\frac{1}{x^{1/3}}\right)^2 > 0$

4 Define $f: \mathbb{R} \to \mathbb{R}$ via $f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Picture:



It may be shown that $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$ if $x \neq 0$

So $f'(x)$ exists if $x \neq 0$ & $\therefore f$ is continuous at each $x \neq 0$.

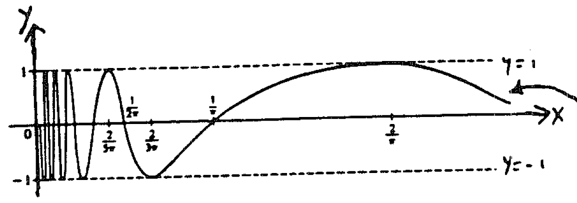
Moreover $|f(x) - f(0)| = \left|x \sin\left(\frac{1}{x}\right) - 0\right| \leq |x|$ so $\forall$ given $\varepsilon > 0$ taking $\delta = \varepsilon$ yields $|f(x) - f(0)| < \varepsilon$ whenever $0 < |x - 0| < \delta$

$\therefore f$ is continuous at $x = 0$, too, so $f: \mathbb{R} \to \mathbb{R}$ is everywhere continuous. But $f'(0)$ does not exist since

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

But $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist. $\therefore f'(0)$ does not exist because $\forall\, \delta > 0$ the graph of f oscillates infinitely often in the interval $(-\delta, \delta)$.

Picture:



We've just drawn the graph of $g(x) = \sin\left(\frac{1}{x}\right)$ for $x > 0$. But this function has a graph that is symmetric about the y-axis