$(F\cdot S)' = F\cdot S' + S\cdot F' = F'\cdot S + F\cdot S'$

$\left(\frac{T}{B}\right)' = \frac{B\cdot T' - T\cdot B'}{B^2}$

$(J^c)' = c\cdot J^{c-1}\cdot J'$

$(F+S)' = F' + S'$

$(c\cdot F)' = c\cdot F'$

$c' = 0$

$f(x) = e^x := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$f'(x) = 0 + 1 + \frac{\cancel{2}x}{\cancel{2}} + \frac{\cancel{3}x^2}{\cancel{3}\cdot 2!} + \frac{\cancel{4}x^3}{\cancel{4}\cdot 3!} + \frac{\cancel{5}\cdot x^4}{\cancel{5}\cdot 4!} + \cdots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$= e^x = f(x)$

$f(x) = \sin x := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$

$f'(x) = 1 - \frac{\cancel{3}x^2}{\cancel{3}\cdot 2!} + \frac{\cancel{5}x^4}{\cancel{5}\cdot 4!} - \frac{\cancel{7}x^6}{\cancel{7}\cdot 6!} + \frac{\cancel{9}x^8}{\cancel{9}\cdot 8!} - \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

$=: \cos x$

$f(x) = \cos x := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots$

$f'(x) = -\frac{2x}{2} + \frac{4x^3}{4\cdot 3!} - \frac{6x^5}{6\cdot 5!} + \frac{8x^7}{8\cdot 7!} - \frac{10x^9}{10\cdot 9!} + \cdots$

$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots$

$= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right)$

$= -\sin x$

2

$(\sin)' = s' = c = \cos$

$(\cos)' = c' = -s = -\sin$

$(\tan)' = t' = \left(\frac{s}{c}\right)' = \frac{c \cdot s' - s \cdot c'}{c^2} = \frac{c \cdot c - s(-s)}{c^2} = \frac{c^2 + s^2}{c^2}$

$= \frac{c^2+s^2}{c^2} = \frac{1^2}{c^2} = \left(\frac{1}{c}\right)^2 = (\sec)^2$

$(\cot)' = \left(\frac{1}{\tan}\right)' = \left(\frac{1}{t}\right)' = \left(t^{-1}\right)' = (-1)\, t^{-2} \cdot t'$

$= -1 \cdot \frac{1}{t^2} \cdot (\sec)^2 = -1 \cdot \frac{1}{\left(\frac{s}{c}\right)^2} \cdot \left(\frac{1}{c}\right)^2$

$= - \frac{1}{\frac{s^2}{c^2}} \cdot \frac{1}{c^2} = - \frac{1}{s^2} = - \frac{1^2}{s^2} = -\left(\frac{1}{s}\right)^2$

$= -(\csc)^2$

$\boxed{\text{or}}$

$(\cot)' = \left(\frac{1}{t}\right)' = \left(\frac{1}{\frac{s}{c}}\right)' = \left(\frac{1}{\frac{s}{c}} \cdot \frac{c}{c}\right)' = \left(\frac{c}{s}\right)'$

$= \frac{s \cdot c' - c \cdot s'}{s^2} = \frac{s(-s) - c \cdot c}{s^2}$

$= \frac{-s^2 - c^2}{s^2} = \frac{-(s^2+c^2)}{s^2} = \frac{-1^2}{s^2}$

$= -\left(\frac{1}{s}\right)^2 = -(\csc)^2$

$(\sec)' = \left(\frac{1}{\cos}\right)' = \left(\frac{1}{c}\right)' = \left(c^{-1}\right)' = -1 \cdot c^{-2} \cdot c'$

$= (-1) \cdot c^{-2} \cdot (-s) = (-1) \cdot \frac{1}{c^2} \cdot (-s) = \frac{1}{c} \cdot \frac{s}{c} = (\sec) \cdot (\tan)$

$(\csc)' = \left(\frac{1}{\sin}\right)' = \left(\frac{1}{s}\right)' = \left(s^{-1}\right)' = (-1) \cdot s^{-2} \cdot s' = (-1) \cdot s^{-2} \cdot c$

$= (-1)\, \frac{1}{s^2} \cdot c = (-1)\, \frac{1}{s} \cdot \frac{c}{s} = (-1)(\csc) \cdot (\cot)$

$(\sin)' = s' = c = \cos$ $\qquad$ $(\csc)' = \left(\frac{1}{\sin}\right)' = -(\csc)\cdot(\cot)$

$(\cos)' = c' = -s = -\sin$ $\qquad$ $(\sec)' = \left(\frac{1}{\cos}\right)' = (\sec)\cdot(\tan)$

$(\tan)' = (\sec)^2$ $\qquad$ $(\cot)' = \left(\frac{1}{\tan}\right)' = -(\csc)^2$

NOTE: A mnemonic device (= memory aid) for recalling $(\cos)'$, $(\cot)'$, & $(\csc)'$ respectively, from $(\sin)'$, $(\tan)'$, & $(\sec)'$ is the following:

$$\begin{array}{ccccc} f' & = & g \\ \updownarrow & \Downarrow & \updownarrow \\ co(f) & = & (-1)\cdot co(g) \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} f' & = & g & \cdot & h \\ \updownarrow & & \updownarrow & & \updownarrow \\ [co(f)]' & = & (-1)\, co(g) & \cdot & co(h) \end{array}$$

i.e., if you have an equation E giving the derivative of one trig function (as another trig function) then you can recall the formula giving the derivative of the co-function of the given trig function by taking each trig function appearing in E, replacing it by its co-function & then multiplying the RHS (= Right Hand Side) of what you obtained by a (-1).

e.g. $\qquad \begin{array}{ccc} s' & = & c \\ \updownarrow & \Downarrow & \updownarrow \\ c' & = & (-1)\, s \end{array} \qquad$ also $\qquad \begin{array}{ccc} c' & = & (-1)\cdot s \\ \updownarrow & & \downarrow \\ s' & = & (-1)(-1)\cdot c = c \end{array}$

$$(\tan)' = (\sec)^2$$
$$\Downarrow$$
$$(\cot)' = (-1)(\csc)^2$$

& also

$$(\cot)' = (-1)(\csc)^2$$
$$\Downarrow$$
$$(\tan)' = (-1)(-1)(\sec)^2 = (\sec)^2$$

Finally,

$$(\sec)' = (\sec)\cdot(\tan)$$
$$\Downarrow$$
$$(\csc)' = (-1)(\csc)\cdot(\cot)$$

& also

$$(\csc)' = (-1)(\csc)(\cot)$$
$$\Downarrow$$
$$(\sec)' = (-1)(\sec)\cdot(\tan)$$

Moral: It suffices to memorize

$$s' = c$$

For then $c' = -s$

Then $(t)' = \left(\frac{s}{c}\right)' = \cdots = (\sec)^2$

Hence $(\cot)' = -(\csc)^2$

Finally, $(\sec)' = \left(\frac{1}{c}\right)' = (c^{-1})' = \cdots = (\sec)\cdot(\tan)$

Then

$$(\csc)' = -(\csc)(\cot).$$

The Chain Rule tells us, that if f' exists at $J(x)$ & if J' exists at x, then $(f \circ J)'$ exists at x &

$$(f \circ J)'(x) = f'[J(x)] \cdot J'(x)$$

From this, we find that

$$\left([J(x)]^c\right)' = c \cdot [J(x)]^{c-1} \cdot J'(x)$$

$$\left(\sin[J(x)]\right)' = \cos[J(x)] \cdot J'(x)$$

$$\left(\cos[J(x)]\right)' = -\sin[J(x)] \cdot J'(x)$$

$$\left(\tan[J(x)]\right)' = (\sec)^2[J(x)] \cdot J'(x)$$

$$\left(\cot[J(x)]\right)' = -(\csc)^2[J(x)] \cdot J'(x)$$

$$\left(\sec[J(x)]\right)' = \sec[J(x)] \cdot \tan[J(x)] \cdot J'(x)$$

$$\left(\csc[J(x)]\right)' = -\csc[J(x)] \cdot \cot[J(x)] \cdot J'(x)$$

$$\left[e^{J(x)}\right]' = e^{J(x)} \cdot J'(x)$$

$f(x) = e^x$ & $g(x) = \log_e(x) = \ln(x)$ are inverse functions. Hence

$$x = f \circ g\,(x) = f[g(x)] = e^{g(x)} = e^{\log_e(x)} = e^{\ln x}$$

&

$$x = g \circ f\,(x) = g[f(x)] = \log_e[f(x)] = \ln[f(x)]$$
$$= \log_e(e^x) = \ln(e^x)$$

i.e.

$$e^{\ln x} = x = e^{\log_e x}$$

&

$$\log_e(e^x) = x = \ln(e^x)$$

From

$x = f \circ g(x)$ we get

$$1 = x' = (f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$
$$= f[g(x)] \cdot g'(x)$$
$$= f \circ g(x) \cdot g'(x)$$
$$= x \cdot g'(x)$$

i.e.,

$$g(x) = \ln(x) = \log_e(x)$$
$$\Downarrow$$
$$g'(x) = [\ln(x)]' = \frac{1}{x}$$

<u>Note</u>: $(-\infty, \infty) \underset{g}{\overset{f}{\rightleftarrows}} (0, \infty)$

So $g(x)$ is <u>not</u> defined if $x = 0$.

Then $\left[\ln(x)\right]' = \dfrac{1}{x}$

$$\Downarrow$$

$$\left(\ln\left[J(x)\right]\right)' = \frac{1}{J(x)} \cdot J'(x) = \frac{J'(x)}{J(x)}$$

Next, recall that

$$\left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \underset{\sin^{-1}}{\overset{\sin}{\rightleftarrows}} [-1, 1]$$

$$y = \sin^{-1} x \longleftarrow\!\shortmid\; x$$

$$y = \sin^{-1}(x)$$

$$\Downarrow$$

$$\sin(y) = x$$

$$\Downarrow$$

$$\cos(y)\,\frac{dy}{dx} = 1$$

$$\Downarrow$$

$$\frac{d\left[\sin^{-1}(x)\right]}{dx} = \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

$$s^2 + c^2 = 1$$

$$\Downarrow$$

$$c^2 = 1 - s^2$$

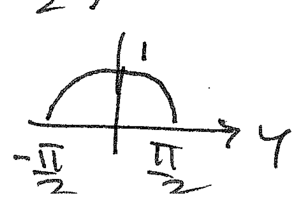
$$\Downarrow$$

$$c = |c| = \sqrt{(c^2)} = \sqrt{1-s^2}$$

$c > 0$

since ~~But~~ $x \in (-1, 1) \Rightarrow y = \sin^{-1}(x) \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$

& for $y \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$, $\cos y > 0$:



Then $y = \sin^{-1}(u)$

$$\Downarrow$$

$$\frac{d[\sin^{-1}u]}{dx} = \frac{dy}{dx} = \frac{d[\sin^{-1}u]}{dx} = \frac{d[\sin^{-1}(u)]}{du}\frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

Similarly,

$$\begin{array}{ccc} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) & \underset{\tan^{-1}}{\overset{\tan}{\rightleftarrows}} & (-\infty, \infty) \\ \in & & \Downarrow \\ y = \tan^{-1}x & \longleftarrow & X \end{array}$$

$$y = \tan^{-1}x$$

$$\tan y = x$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{d[\tan^{-1}(x)]}{dx} = \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$s^2 + c^2 = 1^2$$

$$\Downarrow$$

$$\frac{s^2+c^2}{c^2} = \frac{1^2}{c^2} = \left(\frac{1}{c}\right)^2 = (\sec)^2$$

$$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \left(\frac{s}{c}\right)^2 + 1 = (\tan)^2 + 1$$

Then $y = \tan^{-1}(u)$

$$\Downarrow$$

$$\frac{d[\tan^{-1}(u)]}{dx} = \frac{dy}{dx} = \frac{d\tan^{-1}(u)}{dx} = \frac{d[\tan^{-1}(u)]}{du}\frac{du}{dx}$$

$$= \frac{1}{1+u^2}\frac{du}{dx}$$

Finally

$$(f^g)' = \left[e^{\ln(f^g)}\right]' = \left[e^{g\cdot\ln f}\right]' = e^{g\cdot\ln f}\cdot(g\cdot\ln f)'$$

$$= f^g\cdot\left[g\cdot(\ln f)' + \ln f\cdot g'\right]$$

$$= f^g\left[g\cdot\frac{1}{f}\cdot f' + \ln f\cdot g'\right]$$

$$= f^g\cdot g\cdot f^{-1}f' + f^g\cdot g'\cdot\ln f$$

$$= g f^{g-1}\cdot f' + f^g\cdot g'\cdot\ln f$$

$(f^c)' = c f^{c-1} f'$ since $c' = 0$

$(a^g)' = a^g\cdot g'\cdot\ln a$ since $a' = 0$

$$\ln(A\cdot B) = \ln(A) + \ln(B)$$

$$\ln(A^2) = \ln(A\cdot A) = \ln(A) + \ln(A) = 2\ln(A)$$

$$\ln(A^3) = \cdots = 3\ln(A)$$

$$\ln(A^y) = y\cdot\ln(A)$$

$$\ln\left(\frac{A\cdot B}{C}\right) = \ln(A\cdot B\cdot C^{-1})$$

$$= \ln A + \ln B + \ln(C^{-1})$$

$$= \ln A + \ln B - \ln C$$

$$(\sinh)' = \cosh$$

$$(\cosh)' = \sinh$$

$$\left(\sinh[J(x)]\right)' = \cosh[J(x)]\cdot J'(x)$$

$$\left(\cosh[J(x)]\right)' = \sinh[J(x)]\cdot J'(x)$$

Note

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

VS

$$\cos^2 x + \sin^2 x = 1$$