Math 2107

Rolle's Theorem: (Proved by Michel Rolle in 1691)

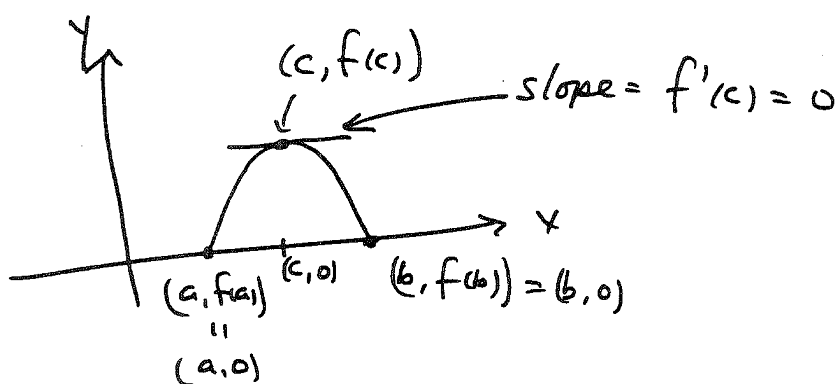
Hypotheses:

1. The function $f : [a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b] := \{x \in \mathbb{R} \mid a \le x \le b\}$.
2. The function $f : [a,b] \to \mathbb{R}$ is differentiable on the open interval $(a,b) := \{x \in \mathbb{R} \mid a < x < b\}$.
3. $f(a) = f(b) = 0$

Conclusion: There is at least one point c with $a < c < b$ for which

$$f'(c) = 0$$

Picture:
(An example)



Proof: By the Extreme Value Theorem with there are points x_m & x_M

$a \le x_m \le b$ & $a \le x_M \le b$

for which

$f(x_m) = m =$ the smallest (or minimum) value of f on $[a,b]$.

& $f(x_M) = M =$ the largest (or Maximum) value of f on $[a,b]$.

Case 1: $m = M = 0$

In this case $f(x) = 0$ for all x in $[a,b]$ so F is constant on $[a,b]$, so $f'(x) = 0$ for any x in $[a,b]$. In this case any c in $[a,b]$ with $a<c<b$ will work.

For example take
$$c = a + \frac{b-a}{2} = \frac{a+b}{2} = \text{the midpoint of the interval } [a,b]$$

Then $a<c<b$ & $f'(c) = 0$.

Case 2: Since $f(a) = f(b) = 0$,

Either $m<0$ or $M>0$

For otherwise $m \geq 0$ and $M \leq 0$

so $0 \leq m \leq M \leq 0$

& $\therefore$ $m = M = 0$ & we're in case 1!

Sub case (i): If $M>0$, then
$$f(x_M) = M > 0$$

Now $f(x_M) = M =$ the maximum value of f on $[a,b]$

Thus $f(x_M) \geq f(x_M + h)$ for all h for which $x_M + h$ lies in the interval $[a,b]$.

i.e. $a \leq x_M + h \leq b$.

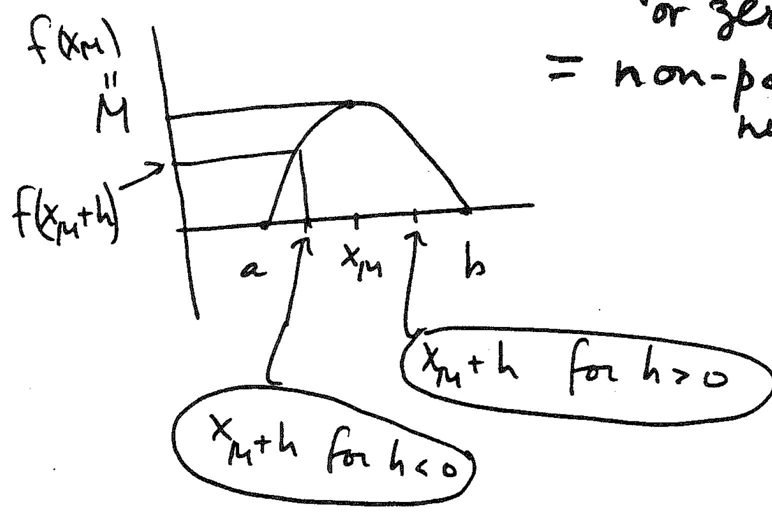
Now
$$f'(x_M) = \underset{h\to 0}{L} \frac{f(x_M+h)-f(x_M)}{h}$$

By assumption
$$f(x_M) \geq f(x_M+h)$$

so
$$f(x_M+h) - f(x_M) = \text{smaller} - \text{larger}$$
$= \text{negative number or zero}$

$= \text{non-positive number}$

Picture:



Hence
$$f'_+(x_M) = \underset{\substack{h\to 0\\ h>0}}{L} \frac{f(x_M+h)-f(x_M)}{h}$$
$$= \text{Lim} \frac{\text{neg.}}{\text{pos.}} = \text{Lim neg.} \leq 0$$

while
$$f'_-(x_M) = \underset{\substack{h\to 0\\ h<0}}{L} \frac{f(x_M+h)-f(x_M)}{h}$$
$$= \text{Lim} \frac{\text{neg}}{\text{neg}} = \text{Lim pos.} \geq 0$$

Now, by assumption, $f'(x)$ exists whenever $a < x < b$.

Moreover $f(x_M) = M > 0$

while, again by assumption,

$f(a) = 0$ <u>and</u> $f(b) = 0$

Thus $x_M \neq a$ and $x_M \neq b$

But $a \leq x_M \leq b$

So $a < x_M < b$

so $f'(x_M)$ exists. But

$f'(x_M)$ exists $\Leftrightarrow$ $f'_+(x_M)$ exists <u>and</u> $f'_-(x_M)$ exists <u>and</u> $f'(x_M) = f'_+(x_M) = f'_-(x_M)$.

Thus, by the results on page 3,

$$0 \leq f'_-(x_M) = f'(x_M) = f'_+(x_M) \leq 0$$

we conclude that $0 \leq f'(x_M) \leq 0$

& hence that $f'(x_M) = 0$

In this case, we may take $c = x_M$. Then $a < c < b$ & $f'(c) = 0$.

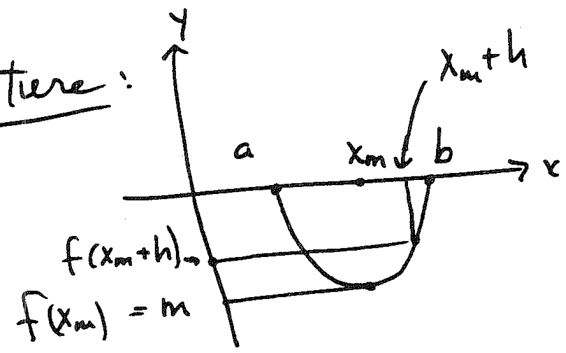
Subcase (ii): If $m < 0$, then

$$f(x_m) = m < 0$$

Now $f(x_m) = m =$ the minimum value of f on $[a,b]$

so $$f(x_m) \le f(x_m + h)$$ for all h for which $$a \le x_m + h \le b$$

Picture:



As before, $a < x_m < b$
because $f(a) = 0 = f(b)$
while $f(x_m) = m < 0$
so $x_m \ne a$ & $x_m \ne b$
but $a \le x_m \le b$, so $a < x_m < b$

Thus,
$$f(x_m + h) - f(x_m) = \text{bigger} - \text{smaller}$$
$=$ positive number or zero
$=$ non-negative number

Then
$$f'_+(x_m) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(x_m+h) - f(x_m)}{h}$$
$$= \text{Lim } \frac{\text{pos.}}{\text{pos.}} = \text{Lim pos.} \geq 0.$$

while
$$f'_-(x_m) = \lim_{\substack{h \to 0 \\ h < 0}} \frac{f(x_m+h) - f(x_m)}{h}$$
$$= \text{Lim } \frac{\text{pos.}}{\text{neg.}} = \text{Lim neg.} \le 0.$$

But $f'(x_m)$ exists since $a < x_m < b$,

so

$$0 \le f'_+(x_m) = f'(x_m) = f'_-(x_m) \le 0$$

so $f'(x_m) = 0$.

In this case, we may take $c = x_m$.

Then $a < c < b$ & $f'(c) = 0$.

The Mean Value Theorem (M.V.T.)

Hypotheses:

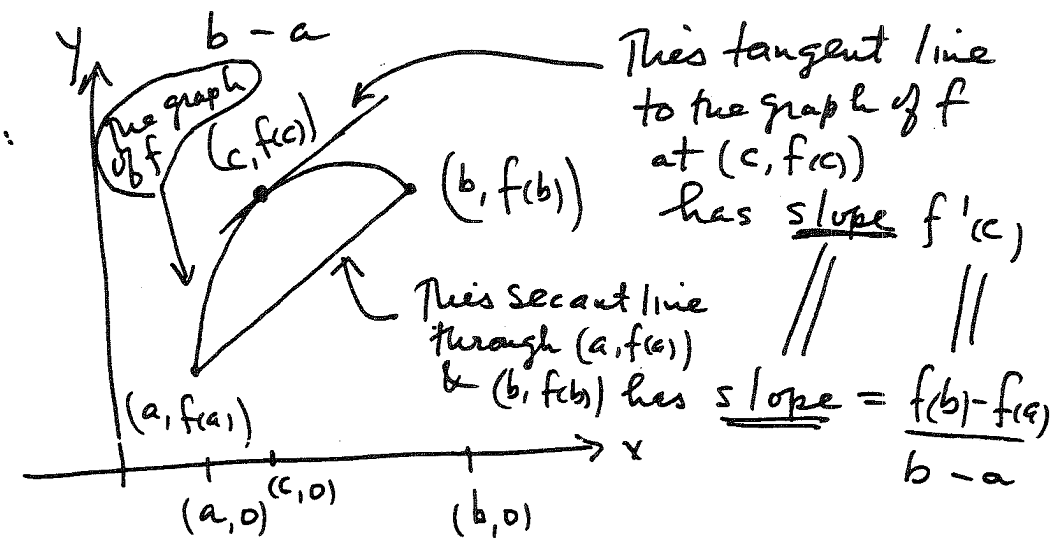
① The function $f : [a,b] \to \mathbb{R}$ is continuous on the closed interval $[a,b]$.

② The function $f : [a,b] \to \mathbb{R}$ is differentiable on the open interval (a,b).

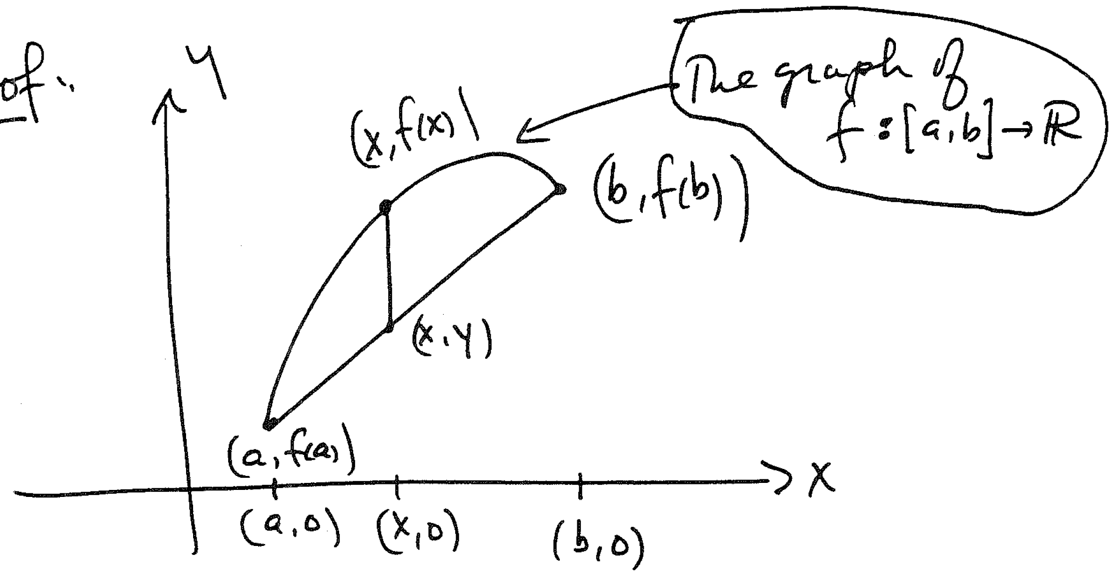
Conclusion: There is at least one point c with $a < c < b$ for which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Picture:



Proof:



The equation of the straight line joining $(a, f(a))$ & $(b, f(b))$ $\longrightarrow$ $\dfrac{y - f(a)}{x - a} = \dfrac{f(b) - f(a)}{b - a}$

$$y = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a)$$

Thus

$d(x) = f(x) - y$ = the <u>vertical distance</u> between (x, y) on the chord (joining $(a, f(a))$ and $(b, f(b))$) and $(x, f(x))$ on the graph of f

$$= f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a)$$

Note: Since

$$d(x) := f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a) - f(a)$$

clearly

$$d(a) = f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](a - a) - f(a)$$

$$= f(a) - (\quad)\cdot 0 - f(a)$$

$$= 0$$

while

$$d(b) = f(b) - \left[\frac{f(b) - f(a)}{\cancel{b - a}}\right]\cancel{(b - a)} - f(a)$$

$$= f(b) - f(b) + f(a) - f(a) = 0$$

Now: $d : [a,b] \to \mathbb{R}$ is defined and continuous on the closed interval $[a,b]$ and $d : [a,b] \to \mathbb{R}$ is differentiable on the open interval (a,b). Moreover

$$d(a) = d(b) = 0$$

HENCE, d satisfies the hypotheses of Rolle's Theorem, so we may conclude that there is at least one point c with $a < c < b$ for which $d'(c) = 0$.

But
$$d(x) := f(x) - \left[\frac{f(b)-f(a)}{b-a}\right](x-a) - f(a)$$

So
$$d'(x) = f'(x) - \left[\frac{f(b)-f(a)}{b-a}\right].(1-0) - 0$$

$$= f'(x) - \frac{f(b)-f(a)}{b-a}$$

Thus
$$0 = d'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

So
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

as was to be shown.

Note: We've just seen that
Rolle's Theorem $\Longrightarrow$ The M.V.T.

But Clearly
The M.V.T. $\Longrightarrow$ Rolle's Theorem

Since if f satisfies the assumptions of Rolle's Theorem then f satisfies the assumptions of the M.V.T. & so there is at least one point c with $a<c<b$ for which $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$.

Clearly, Rolle's Theorem can be weakened to require

$f(b) = f(a)$

(and not necessarily that $f(b) = f(a) \equiv 0$).

Thus, though Rolle's Theorem is a special case of the MVT we can in fact prove the more general result (namely, the MVT) by means of a particular result (namely, Rolle's Theorem). Isn't that beautiful!

Corollary: If the function $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then

(1) $f' \equiv 0$ on $(a,b) \Rightarrow f \equiv$ constant on $[a,b]$

(2) $f' > 0$ on $(a,b) \Rightarrow f \uparrow$ on $[a,b]$
(f is strictly increasing)

(3) $f' < 0$ on $(a,b) \Rightarrow f \downarrow$ on $[a,b]$
(f is strictly decreasing)

Proof:

Case (1) Take x_1 & x_2 to be any 2 points of $[a,b]$ with $a \le x_1 < x_2 \le b$.

By the MVT applied to $[x_1, x_2]$ there is at least one point c with $x_1 < c < x_2$ for which

$$\frac{\text{numerator}}{\text{denominator}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0$$

$\therefore$ the numerator $= 0$

i.e., $f(x_2) - f(x_1) = 0$

so $f(x_1) = f(x_2)$ for any 2 points x_1 & x_2 of $[a,b]$

This proves that f is constant on $[a,b]$

Case (2) Similarly, with the same set-up as in case (1),

$$\frac{?}{\text{pos.}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0$$

Thus

$$\frac{?}{\text{pos.}} = \text{pos.}$$

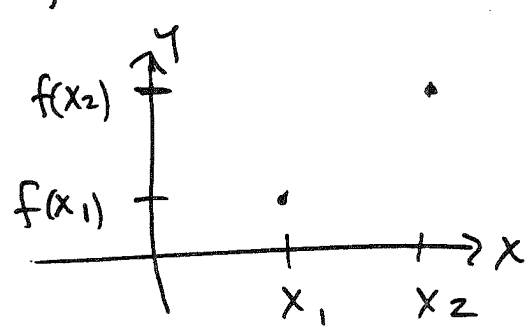
so $f(x_2) - f(x_1) = ? = \text{pos.} > 0$

so $f(x_2) > f(x_1)$.

$\therefore \quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

so $f \uparrow$

Picture:



Case 3: Similarly, with the same set up as in cases 1 & 2

$$\frac{?}{\text{pos.}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) < 0$$

So

$$\frac{?}{\text{pos.}} = \text{neg.}$$

so

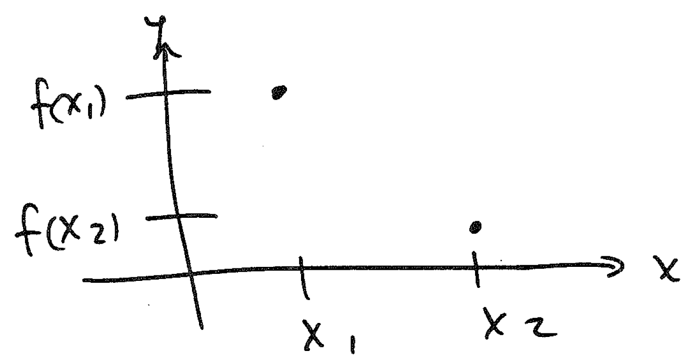
$$f(x_2) - f(x_1) = ? = \text{neg.} < 0$$

so $\quad f(x_2) < f(x_1)$.

$\therefore \quad x_1 < x_2 \implies f(x_1) > f(x_2)$

so $\quad f \searrow$

Picture:

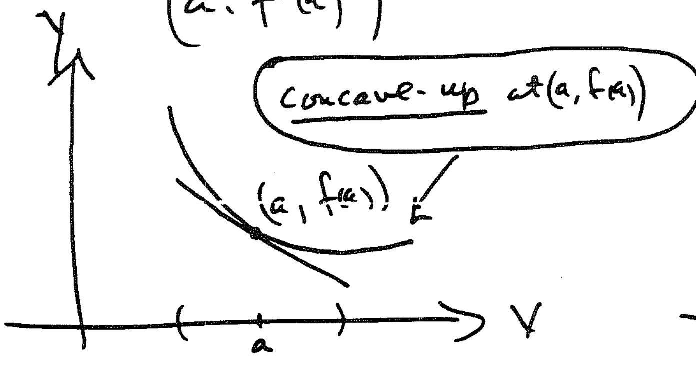


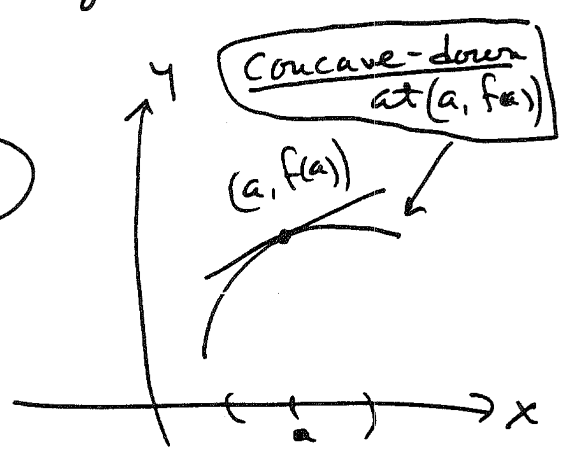
Definition: The graph of a function f is said to be concave up at the point $(a, f(a))$ on the graph of f if $f'(a)$ exists and if there exists an open interval I_a about a such that for all x in I_a with $x \neq a$, the point $(x, f(x))$ on the graph of f lies above the Tangent line to the graph at $(a, f(a))$.

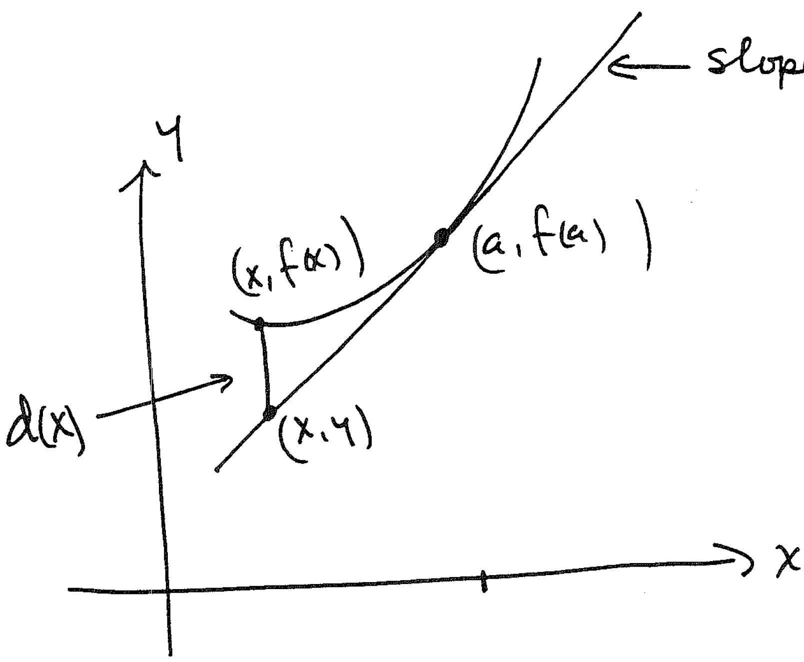
The graph of f is <u>concave-down</u> at $(a, f(a))$ if there is an open interval I_a about a such that if x is in I_a and $x \neq a$, then $(x, f(x))$ lies below the straight line tangent to the graph of f through $(a, f(a))$.

<u>More simply</u>, the graph of a function f is said to be <u>concave-up</u> at a point $(a, f(a))$ if there is a deleted neighborhood D_a of a such that the graph of f restricted to D_a is <u>above</u> the tangent line at $(a, f(a))$.

The graph of f is concave down at $(a, f(a))$ if there is a deleted neighborhood D_a of a such that the graph of f restricted to D_a is <u>below</u> the tangent line at $(a, f(a))$







$$\frac{y - f(a)}{x - a} = f'(a)$$

$$\therefore \quad y = f(a) + f'(a)\cdot(x-a)$$

<u>Define</u>: $d(x) := f(x) - y := f(x) - f(a) - f'(a)\cdot(x-a)$
$\qquad ||$
The directed vertical distance between the point $(x, f(x))$ on the graph of f and the point (x, y) on the straight line tangent to the graph of f through the point $(a, f(a))$.

<u>Clearly</u>, The graph of f is <u>concave-up</u> at a if there is an <u>open interval</u> I_a about a such that whenever x is in I_a but $x \neq a$ then $d(x) > 0$ whereas the graph of f is <u>concave-down</u> at a if there is an open interval I_a about a so that $d(x) < 0$ for all $x \in I_a - \{a\}$.

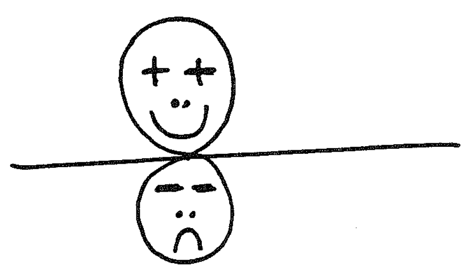
Finally, the graph of a function is concave-up or concave-down on an open interval provided it is concave up or down at each point of that interval.

TEST FOR CONCAVITY

1. Assume f'' exists on an open interval I. If $f''(x) > 0$ for all $x \in I$, then the graph of f is <u>concave-up</u> on I.

2. Assume f'' exists on an open interval I. If $f''(x) < 0$ for all $x \in I$, then the graph of f is <u>concave-down</u> on I.

Picture;



Here "$\cdot\cdot$" stands for the 2nd derivative i.e. $\ddot{x}(t) = \dfrac{d^2x}{dt^2}$

The cartoon  reminds us that if the 2nd derivative is positive, as indicated by ++, then the face has a smile - which is concave up where as if the 2nd derivative face is negative, as indicated by – –, then the face wears a <u>frown</u> which is <u>concave down</u>.

Proof of ①

Recall that the directed distance, $d(x)$, is given by the formula

$$d(x) = f(x) - y = f(x) - f(a) - f'(a)\cdot(x-a).$$

If the graph of f is concave-up at $(a, f(a))$, then there is an open interval I_a about a such that whenever x lies in I_a but is different from a

then

$$d(x) > 0$$

i.e. $d(x) > 0$ for all $x \in I_a - \{a\}$.

<u>Case 1</u>: Suppose $x > a$.

By the MVT applied to the interval $[a, x]$,

$$\frac{f(x) - f(a)}{x - a} = f'(c) \quad \text{for some } c \text{ with } a < c < x$$

$$\therefore \quad f(x) - f(a) = f'(c)\cdot(x-a)$$

so

$$\begin{aligned} d(x) &= f(x) - f(a) - f'(a)\cdot(x-a) \\ &= f'(c)\cdot(x-a) - f'(a)\cdot(x-a) \\ &= [f'(c) - f'(a)]\cdot(x-a) \end{aligned}$$

But $(f')' = f'' > 0$ on I_a

So $f' \uparrow$ on I_a

$\therefore$ $a < c \implies f'(a) < f'(c)$

$\implies f'(c) - f'(a) =$ bigger − smaller

$=$ positive > 0

Also $a < x \implies x - a > 0$

$\therefore$ $d(x) = [f'(c) - f'(a)] \cdot (x - a)$

$= (\text{pos.}) \cdot (\text{pos.}) = \text{pos.} > 0$

$\boxed{\text{Case 2}}$

Similarly, if $x < a$, by the MVT applied to the interval $[x, a]$

$$\frac{f(a) - f(x)}{a - x} = f'(c) \quad \text{for some } c \text{ with } x < c < a$$

But $f'(c) = \dfrac{f(a) - f(x)}{a - x} = \dfrac{(-1)\cdot[f(a) - f(x)]}{(-1)[a - x]} = \dfrac{f(x) - f(a)}{x - a}$

So $f(x) - f(a) = f'(c) \cdot (x - a)$

$\therefore$ $d(x) = f(x) - f(a) - f'(a) \cdot (x - a)$

$= f'(c) \cdot (x - a) - f'(a) \cdot (x - a)$

$= [f'(c) - f'(a)] \cdot (x - a)$

Again,

$(f')' = f'' > 0$ on $I_a \Rightarrow f' \nearrow$ on I_a

$\therefore \quad c < a \Rightarrow f'(c) < f'(a)$

$\Rightarrow f'(c) - f'(a) =$ smaller - larger

$=$ negative < 0

Further $\quad x < a \Rightarrow x - a =$ smaller - bigger

$=$ negative < 0

Thus

$$d(x) = [f'(c) - f'(a)] \cdot (x-a)$$

$$= (\text{neg.}) \cdot (\text{neg.}) = \text{pos.} > 0$$

$\boxed{\text{Proof of (2)}}$ Now suppose that $f'' < 0$ on I_a.

Then $(f')' = f'' < 0$ on I_a

So $f' \searrow$ on I_a.

<u>Case 1</u> If $a < x$, then

$$d(x) = f(x) - f(a) - f'(a) \cdot (x-a)$$

Apply the MVT to the interval $[a, x]$.

Then there is at least one point c with $a < c < x$ for which $\dfrac{f(x) - f(a)}{x - a} = f'(c)$.

Then
$$f(x) - f(a) = f'(c)\cdot(x-a)$$

so
$$\begin{aligned} d(x) &= f(x) - f(a) - f'(a)\cdot(x-a) \\ &= f'(c)\cdot(x-a) - f'(a)\cdot(x-a) \\ &= [f'(c) - f'(a)]\cdot[x-a] \end{aligned}$$

Since $f' \searrow$ on I_a

$$\begin{aligned} a < c \quad &\Rightarrow f'(a) > f'(c) \\ &\Rightarrow f'(c) - f'(a) = \text{smaller} - \text{bigger} \\ &\qquad = \text{negative} < 0 \end{aligned}$$

Further $\quad a < x \Rightarrow x - a = \text{bigger} - \text{smaller}$
$= \text{positive}.$

Thus
$$\begin{aligned} d(x) &= [f'(c) - f'(a)]\cdot[x-a] \\ &= (\text{neg.})\cdot(\text{pos.}) = \text{neg.} < 0 \end{aligned}$$

$\boxed{\text{Case 2}}$ Likewise, if $x < a$, apply the MVT to the interval $[x, a]$ to obtain at least one point c with $x < c < a$ for which

$$\frac{f(a) - f(x)}{a - x} = f'(c)$$

Then

$$f'(c) = \frac{f(a) - f(x)}{a - x} = \frac{(-1)\cdot[f(a)-f(x)]}{(-1)\cdot[a-x]} = \frac{f(x)-f(a)}{x-a}$$

So $\quad f(x) - f(a) = f'(c)\cdot(x-a)$

Then

$$\begin{aligned} d(x) &= f(x) - f(a) - f'(a)\cdot(x-a) \\ &= f'(c)\cdot(x-a) - f'(a)\cdot(x-a) \\ &= [f'(c) - f'(a)]\cdot(x-a) \end{aligned}$$

But $(f')' = f'' < 0 \Rightarrow f' \searrow$ on I_a

so $\quad c < a \Rightarrow f'(c) > f'(a)$

$\Rightarrow f'(c) - f'(a) =$ bigger - smaller $=$ positive > 0

while $\quad x < a \Rightarrow x - a =$ smaller - bigger $=$ negative < 0

$$\therefore d(x) = [f'(c) - f'(a)]\cdot(x-a) = (\text{pos.})\cdot(\text{neg.}) = \text{neg.} < 0$$

This concludes the proof.

<u>Definition</u>: A point $(a, f(a))$ is an <u>inflection point</u> (or <u>point of inflection</u>) of the graph of f if there is a number $\delta > 0$ so that

$f''(x) > 0$ for each x in $(a-\delta, a)$

while

$f''(x) < 0$ for each x in $(a, a+\delta)$

OR $f''(x) < 0$ for each x in $(a-\delta, a)$

while $f''(x) > 0$ for each x in $(a, a+\delta)$

i.e., the graph of f is concave up to the left of a & concave down to the right of a

or the graph of f is concave down to the left of a & concave up to the right of a