2221 Test 1
Tuesday, October 6th, 2009

Name . **Student #**

Instructions. The test runs for one hour - 1:30 pm to 2:30 pm - it is closed book, but calculators are allowed. Show all your work on separate pages and indicate clearly where each question is answered. **Write your name on each page**. Partial credit is given for working, full credit is given for correct answers with justification, but no credit is given without working. There are 4 problems, each worth 5 points. If you have any questions, come and ask.

1. Solve the initial value problem

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{3}{x^3}y = x\cos x, y(\pi/2) = 0.$$

2. Find the general solution to the Bernoulli equation

$$y' = \frac{2y}{x} - x^2y^2$$

3. Find two linearly independent solutions to the equation

$$y'' + 9y = 0$$

Prove they are linearly independent by using the Wronskian function, or directly. Then solve the equation, given data $y(0) = 1, y'(0) = 0$.

4. Obtain the solution to the initial value problem

$$y'' - y = e^x, y(0) = y'(0) = 0.$$

This box is for the grader's use only - do not write answers here.

1. .... 4½ ....
2. .... 0 ....
3. .... 2 ....
4. .... 1 ....

T. .... 7½ ....

#1 $\frac{1}{x^2}\frac{dy}{dx} - \frac{3}{x^3} y = x\cos x \qquad y(\pi/2) = 0$

$$\frac{dy}{dx} - \frac{3}{x} y = x^3 \cos x$$

$$\mu = e^{-3\int \frac{1}{x}dx} = e^{-3\ln|x|} = x^{-3}$$

$$y = \frac{1}{x^{-3}}\left(\int x^{-3}(x^3 \cos x) + C\right) = x^3\left(\int \cos x + C\right)$$

$$y = x^3(\sin x + C) \Rightarrow y = x^3 \sin x + C x^3$$

$$0 = \left(\frac{\pi}{2}\right)^3 \sin\frac{\pi}{2} + C\left(\frac{\pi}{2}\right)^3$$

$$C\frac{\cancel{\pi^3}}{\cancel{8}} = -\frac{\cancel{\pi^3}}{\cancel{8}} \sin\frac{\pi}{2} \Rightarrow C = (-)\sin\frac{\pi}{2}$$

$$\boxed{y = x^3 \sin x - x^3 \underbrace{\sin \pi/2}_{=?}}$$

$4\frac{1}{2}$

#2 $y' = \frac{2y}{x} - x^2y^2$

$y = v^\alpha$; $y' = \alpha v^{\alpha-1} w'$

$\alpha v^{\alpha-1} v' = \frac{2}{x} v^\alpha - x^2 v^{2\alpha}$

$v' = \frac{1}{\alpha}\left(\frac{2}{x}v - x^2 v^{\alpha+1}\right)$

$\Rightarrow y = \frac{1}{v} =$

$v' = -\frac{2}{x}v + x^2$

$v' + \frac{2}{x}v = x^2$

$\mu = x^2$

$\frac{d}{dx}(x^2 v) = x^4$

$x^2 v \quad \frac{-x^5}{5} + c$

$\frac{1}{y} = \frac{x^3}{5} + \frac{c}{x^2} = \frac{x^5 + 5c}{5x^2}$

$y = \frac{5x^2}{x^5+d}$

$\frac{dy}{dx} + P(x)y = Q(x)y^n$

$y' - 2x^{-1}y = -1x^2y^2$

$n=2$, $P(x) = -2/x$; $Q(x) = -x^2$

$y^{-2}\frac{dy}{dx} - 2x^{-1}y^{-1} = -x^2$

let $z = y^{-1} \Rightarrow \frac{dz}{dx} = (-)y^{-2}\frac{dy}{dx}$

$\Rightarrow (-)\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$

$-\frac{dz}{dx} - 2x^{-1}z = -x^2$

$\frac{dz}{dx} + \frac{2z}{x} = x^2$

$\mu = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$z = \frac{1}{x^2}\int x^2 \cdot x^2\,dx$

$z = x^{-2}\left(\frac{1}{5}x^5 + Cx^2\right)$

$x^2 z = \frac{1}{5}x^5 + C$

replace $z = \frac{1}{y}$

$\frac{x^2}{y} = \frac{1}{5}x^5 + C$

$5x^2 = (x^5 + C)y$

$y = \frac{5x^2}{x^5+c}$

#3) $y(0)=1 \quad y'(0)=0$

$y'' + 9y = 0$

$y'' + \varnothing y' + 9y = 0$ $\quad a=1,\ b=0,\ c=9$

$r^2 + \varnothing r + 9 = 0$

$r = \dfrac{-0 \pm \sqrt{0-4(1)(9)}}{2} = \dfrac{\pm\sqrt{-36}}{2} = \cancel{\pm 6i}\ 3i$

$y = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

$\alpha = 0$

$\beta = 6$ x

$\boxed{y = A\cos \cancel{6}^{3} t + B\sin \cancel{6}^{3} t}$

$y' = \cancel{6}^{3} A\cos \cancel{6}^{3} t - A\sin \cancel{6}^{3} t + 6B\sin 6t + B\cos 6t \quad (6A + B = 0)$

$y'' = 36A\cos 6t - 6A\sin 6t - 6A\sin 6t - A\cos 6t + 36B\sin 6t + 6B\cos 6t$
$+ 6B\cos 6t - B\sin 6t$

$y'' = 35A\cos 6t - 12A\sin 6t + 35B\sin 6t + 12B\cos 6t$

$y(0) = 9A\cos \cancel{6}^{3} t + 9B\sin 6t$

$1 = 9A\cos 0^{\nearrow 1} + 9B\sin \varnothing^{\nearrow 0}$

W?

$1 = 9A$

$\boxed{A = 1/9}$ x

$6(\tfrac{1}{9}) + B = 0$

$\boxed{B = -2/3}$ x

$\boxed{y = 1/9\cos 6t - 2/3\sin 6t}$

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$y = C_1\cos 3x + C_2\sin 3x$

$y' = (-3C_1\sin 3x + 3c_1\cos 3x$

$c_1 = 1$

#4) $y'' - y = e^{x}$ $\quad y(0)=0$, $y'(0)=0$

$a=1$, $b=0$, $c=(-1)$ $\quad \pm\frac{\sqrt{4(+1)}}{2} = \frac{2}{2} \Rightarrow r = +1 \text{ or } -1$

$y = C_1 e^{t} + C_2 e^{-t}$ ✓

$y' = C_1 e^{t} - C_2 e^{-t}$ $\quad y(0)\, C_1 + C_2 = 0 \Rightarrow \boxed{C_1 = C_2}$

$y'' = C_1 e^{t} + C_2 e^{-t}$ $\quad c_1 = ?\ \ c_2 = ?$

$y(-1) = -C_1 e^{t} - C_2 e^{-t}$

y_p?

$\boxed{y = C(e^{t} + e^{-t})}$

$y_1 = e^{x}, \ y_2 = e^{-x}$

$y_h = c_1 e^{x} + c_2 e^{-x}$

$y_p = A x e^{x}$

$y_p' = A(x+1)e^{x}$

$y''_p = A(x+2)e^{x}$

$y''_p - y_p = 2Ae^{x} \Rightarrow e^{x} \Rightarrow A = \frac{1}{2}$

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