MATH 2221 - EXAM II

July 22, 2008

69+4

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NAME:

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1. Determine the Laplace transform

(a)(5pts)

$$f(t) = \begin{cases} e^{2t}, & 0 < t < 3 \\ 1, & 3 < t \end{cases}$$

$$\int_0^3 e^{-st} e^{2t}\,dt + \int_3^\infty e^{-st}\,dt$$

$$= \left[\frac{-e^{-t(s-2)}}{s-2}\right]_0^3 + \left[\frac{-e^{-ts}}{s}\right]_3^\infty$$

$$= \boxed{\frac{1}{s-2} - \frac{e^{-3(s-2)}}{s-2} + \frac{e^{-3s}}{s}}$$ ✓

(b)(5pts) $g(t) = t^4 e^{5t} - e^t \cos(\sqrt{7}t)$

$4! = 24$ $$\boxed{\frac{24}{(s-5)^5} - \frac{(s-1)}{(s-1)^2+7}}$$ ✓

(c)(5pts) $h(t) = \delta(t-\pi)\sin t$

$$= \sin t\, \delta(t-\pi)$$

$$= e^{-\pi s}\, \mathcal{L}\{\sin(t-\pi)\}$$

$$= e^{-\pi s}\, \mathcal{L}\{-\sin t\}$$

$$= \boxed{e^{-\pi s}\frac{(-1)}{s^2+1}}$$

2

8

(d)(5pts) $k(t) = \int_0^t (t-v)e^{3v}dv$

$$t * e^{3t}$$

$$\frac{1}{s^2} * \frac{1}{s-3} = \boxed{\frac{1}{s^2(s-3)}}$$

2. (8pts) Determine the inverse Laplace transform

$$F(s) = \frac{5s^2+34s+53}{(s+3)^2(s+1)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+1}$$

$$= A(s+3)(s+1) + B(s+1) + C(s+3)^2 = 5s^2 + 34s + 53$$

$s = -1$; $C(4) = 5 - 34 + 53$ $\quad$ $\begin{array}{r} 453 \\ -34 \\ \hline 19 \\ \hline 24 \end{array}$

$C(4) = 24$

$C = 6$

$$A(s^2+4s+3) + B(s+1) + C(s^2+6s+9)$$

$$s^2(A+C) + s(4A+B+6C) + 3A + B + 9C$$

s^2: $5 = A + C = A + 6$; $A = -1$

s : $34 = 4(-1) + B + 6(6)$

$36 - 4 + B = 34$ $\quad B = 2$

$32 + B = 34$

$$\frac{-1}{s+3} + \frac{2}{(s+3)^2} + \frac{6}{s+1} = \boxed{-e^{-3t} + 2te^{-3t} + 6e^{-t}}$$

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3. (18pts) Solve the initial value problem using the method of Laplace transforms

$$y'' + y = t^2 + 2, \quad y(0) = 1, \; y'(0) = -1.$$

$y'' = \left[s^2 Y(s) - s + 1\right] + Y(s) = \frac{2}{s^3} + \frac{2}{s} = \frac{2 + 2s^2}{s^3}$

$Y(s)\left[s^2 + 1\right] - s + 1 = \frac{2 + 2s^2}{s^3}$

$Y(s) = \frac{2 + 2s^2}{s^3(s^2+1)} + \frac{s}{s^2+1} + \frac{1}{s^2+1}$

$\frac{2 + 2s^2}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2+1}$

$2 + 2s^2 = A(s^2)(s^2+1) + B(s)(s^2+1) + C(s^2+1) + (Ds + E)(s^3)$

$s = 0 \; ; \; 2 = C(1) \; ; \; \boxed{C = 2}$

$= s^4(A + D) + s^3(B + E) + s^2(A + C) + s(B) + C$

$s^2 = 2 = A + C \; ; \; \boxed{A = 0}$

$s = 0 = B = 0$

$s^2 + 1$

$+ \cos t \quad + \sin t$

10

4. (18pts) Solve the initial value problem using the method of Laplace transform

$$y'' + 5y' + 6y = tu(t-2), \quad y(0) = 0 \;, \quad y'(0) = 1.$$

$= e^{-2s}(\mathcal{L}\{t+2\})$

$[s^2Y(s) - 1] + 5[sY(s) - 0] + 6Y(s) = e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$

$Y(s)[s^2 + 5s + 6] - 1 = e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$

$Y(s) = \frac{e^{-2s}}{s^2+5s+6}\left(\frac{1+2s}{s^2}\right) + \frac{1}{s^2+5s+6}$

$= e^{-2s}\left(\frac{1+2s}{(s+2)(s+3)(s^2)}\right) + \frac{1}{(s+2)(s+3)}$ $= 1 = A(s+3) + B(s+2)$

$s=-3$ $\quad 1 = B(-1)$

$s=-2$ $\quad A = 1 \quad B = -1$

$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}$

$1+2s = A(s)(s+2)(s+3) + B(s+2)(s+3) + C(s+3)(s^2) + D(s+2)(s^2)$

$s=0 \;=\; 1 = B(6) \quad ; B = \frac{1}{6}$

$A(s^3+5s^2+6s) + B(s^2+5s+6) + C(s^3+3s^2) + D(s^3+2s^2)$

$s^3(A+C+D) + s^2(5A+B+3C+2D) + s(6A+5B) + 6B$

$s: \frac{12}{6} = 6A + \frac{5}{6} = \frac{7}{6} = 6A = \frac{7}{36}$

$s^2 = 0 = \frac{7}{36}(5) + \frac{6}{36} + 3C + 2D$ $\quad = \frac{35}{36} + \frac{6}{36} = \frac{41}{36}$

$s^3 = 0 = \frac{7}{36} + C + D$

$\frac{41}{36} + 3C + 2D = 0$

$-\frac{14}{36} + -2C + 2D = 0$

$-\frac{14}{36} \quad C = -\frac{41}{36}$

$C = \frac{-27}{36}$

$\frac{27}{36} + D = -\frac{7}{36} \qquad \cancel{D = \frac{-34}{36}}$

$\begin{array}{r} 341 \\ -14 \\ \hline 27 \end{array}$

$e^{-2s}\left[\frac{7}{36}\left(\frac{1}{s}\right) + \frac{1}{6}\left(\frac{1}{s^2}\right) + \frac{-27}{36}\left(\frac{1}{s+2}\right) - \frac{34}{36}\left(\frac{1}{s+3}\right)\right]$

5. (18pts) Solve the initial value problem

8+4

$$y'(t) - 2\int_0^t y(v)\sin(t-v)\,dv = 1, \quad y(0) = 1.$$

$$sY(s) - 1 - 2Y(s)\cdot\frac{1}{s^2+1} = \frac{1}{s}$$

$$Y(s) = \left(s + \frac{2}{s^2+1}\right) = \frac{1}{s} + 1$$

$$\frac{(1+s)(s^2+1)}{s(s^3+s^2+2)} = \frac{s^2+1+s^3+s}{s^3+s^2+s+1}$$

$$Y(s)\left(\frac{s^2-2}{s^2+1}\right) Y(s)\frac{s^2+3}{s^2-1} = \frac{1}{s} + 1$$

$$\frac{s^3+s^2+2}{s^2+1} \qquad Y(s) = \frac{s^2+1}{s^2-1}\left(\frac{1}{s}\right) + \frac{s^2+1}{s^2-1}$$

$$Y_s\left[\frac{s(s^2+1)-2}{s^2+1}\right] = \frac{1}{s} + 1 \qquad \frac{s^2+1}{(s+1)(s-1)(s)} + \frac{s^2+1}{s^2-1}$$

$$Y_s\left[\frac{s^3+s-2}{s^2+1}\right] = \frac{1+s}{s}$$

$$Y(s) = \frac{1+s}{s}\cdot\frac{s^2+1}{s^3+s-2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s} \qquad \to \frac{A}{s-1} + \frac{B}{s+1}$$

s^4+

$$s^2+1 = A(s+1) + B(s-1)$$

$s=-1 = 2 = B = -1$

$A = A$

$$s^2+1 = A(s-1)(s) + B(s+1)(s) + C(s+1)(s-1)$$

$s=0 = 1 = \boxed{C=-1}$

$s=-1 = 2 = A(-2)(-)$

$A=1$

$s=1 = 2 = B(2).\ B=1$

$$\frac{1}{s+1} + \frac{1}{s-1} + \frac{-1}{s} + \frac{1}{s-1} + \frac{-1}{s+1} \quad ?$$

$$\boxed{e^{-t} + e^{t} - t + e^{t} + -e^{-t}}$$

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6. (12pts) Consider the initial value problem

$$y'' - 2y' + 5y = g(t), \quad y(0) = 0, \quad y'(0) = 2.$$

(a) Find the transfer function $H(s)$.
(b) Find the impulse response function.

a) $H(s) = \frac{Y(s)}{G(s)} = \frac{1}{s^2-2s+5} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$ ✓

impulse response function?

$y'' - 2y + 5y = 0$

$r = \frac{4 \pm \sqrt{4-20}}{2} = 2 \pm \frac{\sqrt{-16}}{2} = 2 \pm 2i$

$= c_1 e^{2t}\cos 2t + c_2 e^{2t}\sin 2t$

$0 = c_1 \cos 0 \cancel{+ c_2}$

$y' = 2e^{2t}\cos 2t - 2e^{2t}\sin 2t + 2e^{2t}\cos 2t \cancel{+ 2e^{2t}\sin 2t}$

$2 = 2c_1 + 2c_2$

$c_1 = 0$

$c_2 = 1$

$e^{2t}\sin 2t + \mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$