Taylor's Series (First derivative)

(Forward difference method)

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)(\Delta x)^2}{2!} + \frac{f'''(x)(\Delta x)^3}{3!} + \frac{f^{[4]}(x)(\Delta x)^4}{4!} + \cdots\cdots$$

Truncating after the second term

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(Backward difference method)

$$f(x-\Delta x) = f(x) + f'(x)(-\Delta x) + \frac{f''(x)(-\Delta x)^2}{2!} + \frac{f'''(x)(-\Delta x)^3}{3!} + \frac{f^{[4]}(x)(-\Delta x)^4}{4!} + \cdots\cdots$$

Truncating after the second term

$$f(x-\Delta x) \approx f(x) - f'(x)\Delta x$$

$$f'(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

(Central difference method)

$$\frac{\text{Forward difference for } \tfrac{1}{2}\Delta x + \text{Backward difference for } \tfrac{1}{2}\Delta x}{2}$$

$$f'(x) \approx \frac{f(x+\frac{1}{2}\Delta x) - f(x)}{\frac{1}{2}\Delta x}$$ forward diff.

$$f'(x) \approx \frac{f(x) - f(x-\frac{1}{2}\Delta x)}{\frac{1}{2}\Delta x}$$ backward diff.

$$2f'(x) \approx \frac{f(x+\frac{1}{2}\Delta x) - f(x-\frac{1}{2}\Delta x)}{\frac{1}{2}\Delta x}$$

$$f'(x) \approx \frac{f(x+\frac{1}{2}\Delta x) - f(x-\frac{1}{2}\Delta x)}{\Delta x}$$ Central diff.

$$\frac{\text{Forward diff. for } \Delta x + \text{Backward diff for } \Delta x}{2}$$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ and

$$f'(x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

adding

$$2f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{\Delta x}$$

or

$$f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

$$h' = \frac{h_{i+1} - h_{i-1}}{2\Delta x}$$

Taylor's Series (Second derivative)

$$f(x+\Delta x) = f(x) + f'(x)\,\Delta x + f''(x)\frac{(\Delta x)^2}{2!} + \cdots$$

$$f(x-\Delta x) = f(x) + f'(x)(-\Delta x) + f''(x)\frac{(-\Delta x)^2}{2!} + \cdots$$

Adding gives

$$f(x+\Delta x) + f(x-\Delta x) \approx 2f(x) + 2\frac{f''(x)(\Delta x)^2}{2}$$

$$f''(x)(\Delta x)^2 \approx f(x+\Delta x) - 2f(x) + f(x-\Delta x)$$

$$f''(x) \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

Using notation in text

$$h_i'' = \frac{h_{i+1} - 2h_i + h_{i-1}}{(\Delta x)^2}$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \qquad \text{Eqn } (9.1.1)$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j}}{(\Delta x)^2} +$$

$$+ \frac{h_{i,j+1} - 2h_{i,j} + h_{i,j-1}}{(\Delta y)^2}$$

if $\Delta x = \Delta y = \Delta$, then

$$= \frac{h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} - 4h_{i,j}}{\Delta^2} = 0$$

$$h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1} - 4h_{i,j} = 0$$

$$h_{i,j} = \frac{1}{4}\left(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}\right)$$

We solved analytically the equation (for radial flow on a perfectly round island)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad \text{becomes} \quad \frac{\partial^2 h}{\partial z^2} + \frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = 0$$

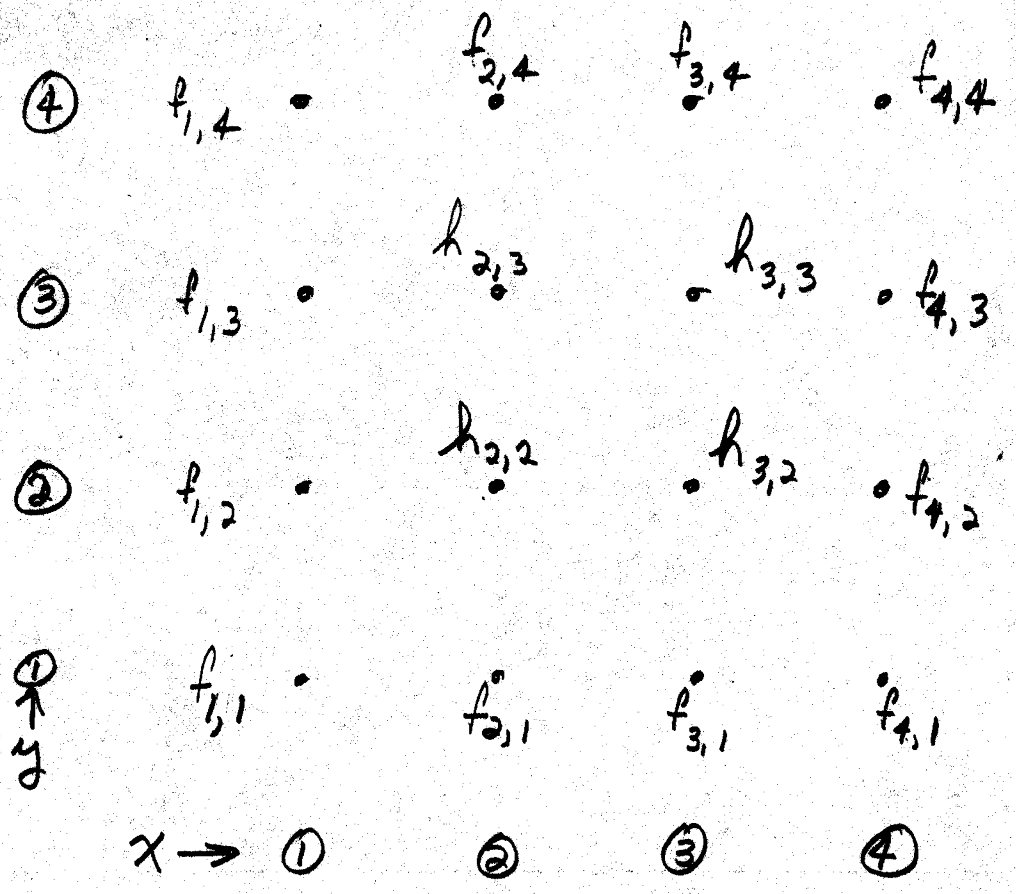
with the boundary condition of

$h = H_0$ at $r = r_0$

The equation we obtained was Dupuit's Eqn. [Notes 52-5

By the finite difference method we can solve the steady flow equation for aquifers that are not circular and also for aquifers that have a varying h for the boundary condition.

For the boundary conditions where h is specified (either $h = f(x,y)$ or $h =$ constant) we have



$$
\left.
\begin{aligned}
h_{2,2} &= \tfrac{1}{4}\left(f_{1,2} + f_{2,1} + h_{3,2} + h_{2,3}\right)\\
h_{3,2} &= \tfrac{1}{4}\left(f_{3,1} + f_{4,2} + h_{3,3} + h_{2,2}\right)\\
h_{2,3} &= \tfrac{1}{4}\left(f_{1,3} + f_{2,4} + h_{3,3} + h_{2,2}\right)\\
h_{3,3} &= \tfrac{1}{4}\left(f_{3,4} + f_{4,3} + h_{3,2} + h_{2,3}\right)
\end{aligned}
\right\}
\text{4 Equations in 4 unknowns}
$$

In general, there will be n linear equations in n unknowns.

For the boundary condition of an impermeable boundary we have

$$Q_n = 0 \Rightarrow \text{since} \left(Q = AV \right)$$

$$V_n = 0 \Rightarrow \text{since} \left(V = -K \frac{\partial h}{\partial L} \right)$$

$$-K \frac{\partial h}{\partial x} = 0 \text{ or } -K \frac{\partial h}{\partial y} = 0 \Rightarrow$$

$$\frac{\partial h}{\partial x} = 0 \quad \text{or} \quad \frac{\partial h}{\partial y} = 0$$

at the boundary of the aquifer.

Therefore from the central difference for the first derivative (when we go $+\Delta x$ and $-\Delta x$ not $+\frac{1}{2}\Delta x$ and $-\frac{1}{2}\Delta x$)

$$\frac{\partial h}{\partial x} \approx \frac{h(x+\Delta x, y) - h(x-\Delta x, y)}{2\Delta x} = 0$$

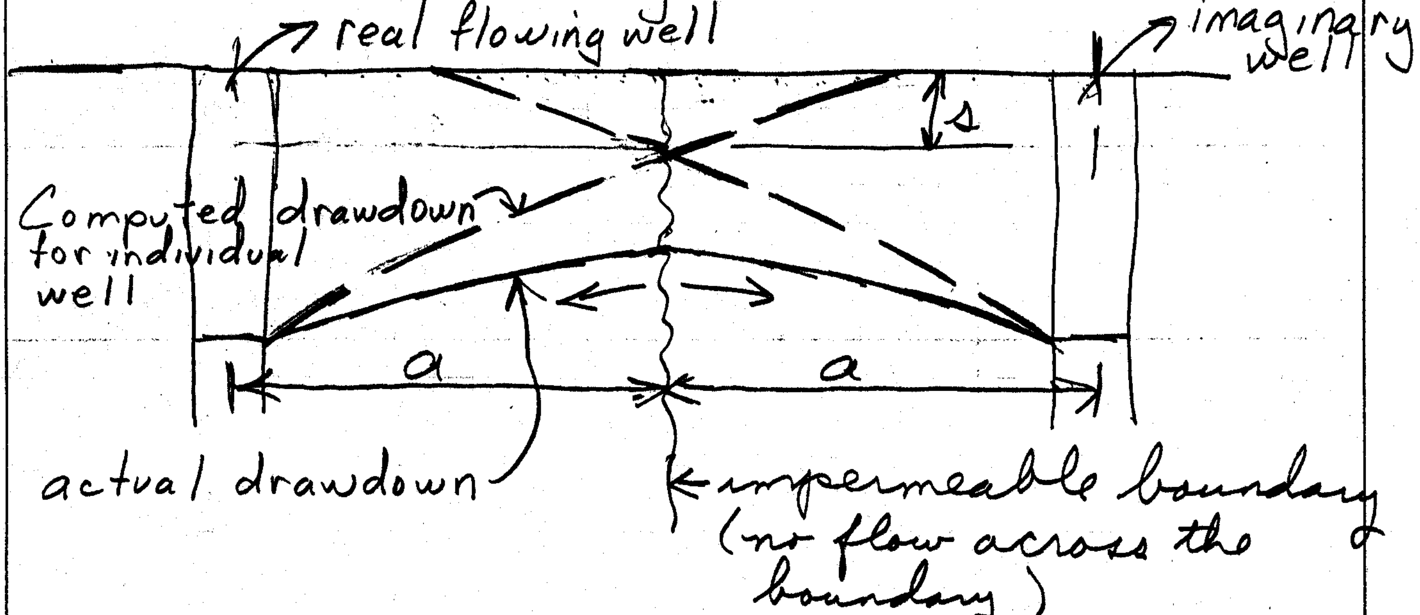
$$\Rightarrow h(x+\Delta x, y) = h(x-\Delta x, y)$$

$$\text{or} \quad h_{i+1,j} = h_{i-1,j}$$

(This is for a vertical impermeable boundary, similarly $h_{i,j+1} = h_{i,j-1}$ for a horizontal impermeable boundary.)

In other words, the value in an imaginary node across the boundary should be equal to the value inside the boundary.

Method of Images



$$s_{actual} = s_{real} + s_{imaginary}$$

Therefore in the general central difference algorithm developed for the steady flow equation

$$h_{i,j} = \frac{1}{4}\left(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}\right)$$

we have

$$h_{i,j} = \frac{1}{4}\left(2h_{i+1,j} + h_{i,j-1} + h_{i,j+1}\right)$$

(this is for the boundary vertically on the "left" side).

and

$$h_{i,j} = \frac{1}{4}\left(2h_{i-1,j} + h_{i,j-1} + h_{i,j+1}\right)$$

(this is for the boundary vertically on the "right" side).

Example

A simple example, for a rectangular region, is shown in Figure 9.3. Along the upper boundary the head is given to be 100. In the lower left corner a zero head is specified. All other boundaries are impermeable. An initial estimate of all unknown values is required in order to solve the problem. For simplicity, these have all been assumed to be equal to the average of the boundary values, i.e., 50.

The first figure shows the initial data. By examining these values it must be concluded that they do not satisfy the conditions (9.1.9). However, they can now be corrected, successively, so that they do satisfy the basic equation. In the central part of the figure, the algorithm (9.1.9) has been applied once to all nodes, starting in the upper left corner, and rounding off to the nearest integer. Of course, subsequent corrections partly destroy the agreement reached in a previous correction, and the new values still do not satisfy (9.1.9). The results show a definite improvement on the initial values, however. After a number of *iterations*, in each of which all values are updated, the correct solution, represented in the right part of the figure, is obtained. Example

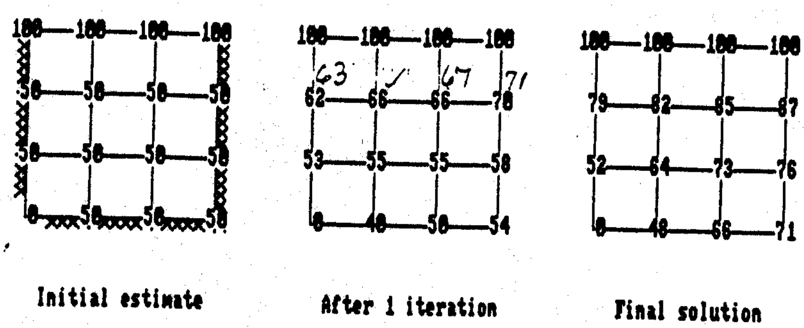


Fig. 9.3. Example of the finite difference method.

Application to example in book (Fig. 9.3)

100	100	100	100
•	•	•	•
•	•	•	•
0	•	•	•

Aquifer with a constant head of 100 at one end and a head of 0 at one corner. All other boundaries are impermeable.

100	100	100	100
50	50	50	50
50	50	50	50
0	50	50	50

100	100	100	100
63	66	67	71
41	52	55	58
0	39	50	54

Iteration "1"

Calculated from top left corner to bottom right.

100	100	100	100
68	72	75	77
43	52	59	62
0	39	53	58

Iteration "2"

Final solution (see book) →

100	100	100	100
79	82	85	87
52	64	73	76
0	48	66	71

Notice that the final solution can be converged on much quicker with a better first approximation.

The method described above is known as the method of *relaxation* after Southwell (1940), who used the term to indicate that in each step one of the errors is relaxed. Experience shows that the method converges for all problems of potential flow. In mathematical terminology, the relaxation method is also known as the *Gauss—Seidel method* (see Appendix A).

Generalize solution for varying Δx + Δy

Before presenting a computer program that will perform the finite difference calculations, the basic algorithm will be generalized to the case of a mesh of varying intervals. This can be done most conveniently by the application of Taylor's series expansion formula. If the value of a function $F(x)$ at the point x_i is denoted by F_i, the values at the neighboring points x_{i+1} and x_{i-1} can be expressed in terms of the values of the function and its derivatives at point x_i, as follows

$$F_{i+1} = F_i + (x_{i+1} - x_i)\frac{dF}{dx} + \tfrac{1}{2}(x_{i+1} - x_i)^2 \frac{d^2F}{dx^2} + \cdots \tag{9.1.11}$$

$$F_{i-1} = F_i + (x_i - x_{i-1})\frac{dF}{dx} + \tfrac{1}{2}(x_i - x_{i-1})^2 \frac{d^2F}{dx^2} + \cdots \tag{9.1.12}$$

Elimination of the first derivative dF/dx from these equations gives

$$\frac{d^2F}{dx^2} = A_iF_{i+1} + B_iF_{i-1} - (A_i + B_i)F_i + \cdots \tag{9.1.13}$$

where

$$A_i = 1/[(x_{i+1} - x_i)(x_{i+1} - x_{i-1})/2], \tag{9.1.14}$$

$$B_i = 1/[(x_i - x_{i-1})(x_{i+1} - x_{i-1})/2]. \tag{9.1.15}$$

Equation (9.1.13) is the generalization of (9.1.7) for variable distances between successive mesh points. If $x_{i+1} - x_i = x_i - x_{i-1} = \Delta x$, Equation (9.1.13) reduces to (9.1.7).

The approximation (9.1.13) can directly be used to approximate the partial derivative $\partial^2F/\partial x^2$, and a similar formula can be derived for the partial derivative $\partial^2F/\partial y^2$. In this way, the following generalization of (9.1.9) is obtained

$$(A_i + B_i + C_j + D_j)\phi_{i,j} = A_i\phi_{i+1,j} + B_i\phi_{i-1,j} + C_j\phi_{i,j+1} + D_j\phi_{i,j-1} \tag{9.1.16}$$

where A_i and B_i are given by (9.1.14) and (9.1.15), and where

$$C_j = 1/[(y_{j+1} - y_j)(y_{j+1} - y_{j-1})/2], \tag{9.1.17}$$

$$D_j = 1/[(y_j - y_{j-1})(y_{j+1} - y_{j-1})/2]. \tag{9.1.18}$$

If all mesh sizes are equal the coefficients A_i, B_i, C_j and D_j are all equal and (9.1.16) reduces to (9.1.9).

A computer program, in BASIC, based on the algorithm (9.1.16) is reproduced as Program BV9-1.

Steady flow by finite difference

```
100 DEFINT I-N:KEY OFF:GOSUB 540
110 PRINT"---  Bear & Verruijt - Groundwater Modeling"
120 PRINT"---  Plane steady groundwater flow"
130 PRINT"---  Finite differences":PRINT"---  Program 9.1"
140 PRINT"---  Rectangular area, with irregular mesh":PRINT:PRINT
150 DIM X(21),Y(21),F(21,21),IP(21,21),A(21),B(21),C(21),D(21)
160 INPUT"Number of mesh lines in x-direction (max. 20) ";NX
170 INPUT"Number of mesh lines in y-direction (max. 20) ";NY:PRINT
180 FOR I=1 TO NX:PRINT"x-coordinate of line ";I;:INPUT X(I):NEXT I
190 PRINT:FOR I=1 TO NY:PRINT"y-coordinate of line ";I;:INPUT Y(I)
200 NEXT I:X(0)=X(1)-(X(2)-X(1)):X(NX+1)=X(NX)+(X(NX)-X(NX-1))
210 Y(0)=Y(1)-(Y(2)-Y(1)):Y(NY+1)=Y(NY)+(Y(NY)-Y(NY-1)):GOSUB 540
220 A=0:K=0:PRINT"The head must be given in at least one point"
230 PRINT:PRINT"  i = ";:INPUT I:PRINT"  j = ";:INPUT J
240 PRINT"  F = ";:INPUT F(I,J):IP(I,J)=1:A=A+F(I,J):K=K+1
250 PRINT:PRINT"Repeat input of given head (Y/N) ? ";:GOSUB 510
260 IF A$="Y" THEN 230
270 A=A/K:FOR I=0 TO NX+1:FOR J=0 TO NY+1:IF IP(I,J)=0 THEN F(I,J)=A
280 NEXT J:NEXT I
290 FOR I=1 TO NX:A(I)=2/((X(I+1)-X(I))*(X(I+1)-X(I-1)))
300 B(I)=2/((X(I)-X(I-1))*(X(I+1)-X(I-1))):NEXT I
310 FOR J=1 TO NY:C(J)=2/((Y(J+1)-Y(J))*(Y(J+1)-Y(J-1)))
320 D(J)=2/((Y(J)-Y(J-1))*(Y(J+1)-Y(J-1))):NEXT J:NI=NX*NY:RX=1.4
330 PRINT:PRINT"Iteration";:FOR IT=1 TO NI:PRINT IT;
340 FOR I=1 TO NX:F(I,0)=F(I,2)
350 FOR J=1 TO NY:F(0,J)=F(2,J):IF IP(I,J)>0 THEN 380
360 A=A(I)*F(I+1,J)+B(I)*F(I-1,J)+C(J)*F(I,J+1)+D(J)*F(I,J-1)
370 A=A/(A(I)+B(I)+C(J)+D(J)):F(I,J)=F(I,J)+RX*(A-F(I,J))
380 F(NX+1,J)=F(NX-1,J):NEXT J:F(I,NY+1)=F(I,NY-1):NEXT I
390 FOR J=1 TO NY:F(NX+1,J)=F(NX-1,J)
400 FOR I=1 TO NX:F(I,NY+1)=F(I,NY-1):IF IP(I,J)>0 THEN 430
410 A=A(I)*F(I+1,J)+B(I)*F(I-1,J)+C(J)*F(I,J+1)+D(J)*F(I,J-1)
420 A=A/(A(I)+B(I)+C(J)+D(J)):F(I,J)=F(I,J)+RX*(A-F(I,J))
430 F(I,0)=F(I,2):NEXT I:F(0,J)=F(2,J):NEXT J
440 NEXT IT:GOSUB 540:PRINT"Output":PRINT:A$="#####.###"
450 FOR I=1 TO NX:FOR J=1 TO NY
460 PRINT" x = ";:PRINT USING A$;X(I);
470 PRINT" -  y = ";:PRINT USING A$;Y(J);
480 PRINT" -  F = ";:PRINT USING A$;F(I,J):NEXT J,I:PRINT
490 PRINT"Repeat iterations (Y/N) ? ";:GOSUB 510:IF A$="Y" THEN 330
500 PRINT:END
510 A$=INPUT$(1):IF A$="Y" OR A$="y" THEN A$="Y":PRINT"Yes":RETURN
520 IF A$="N" OR A$="n" THEN A$="N":PRINT"No":RETURN
530 GOTO 510
540 CLS:LOCATE 1,28,1:COLOR 0,7:PRINT" Finite Differences - 1 ";
550 COLOR 7,0:PRINT:PRINT:RETURN
```

The head ϕ or h is represented by F in the program.

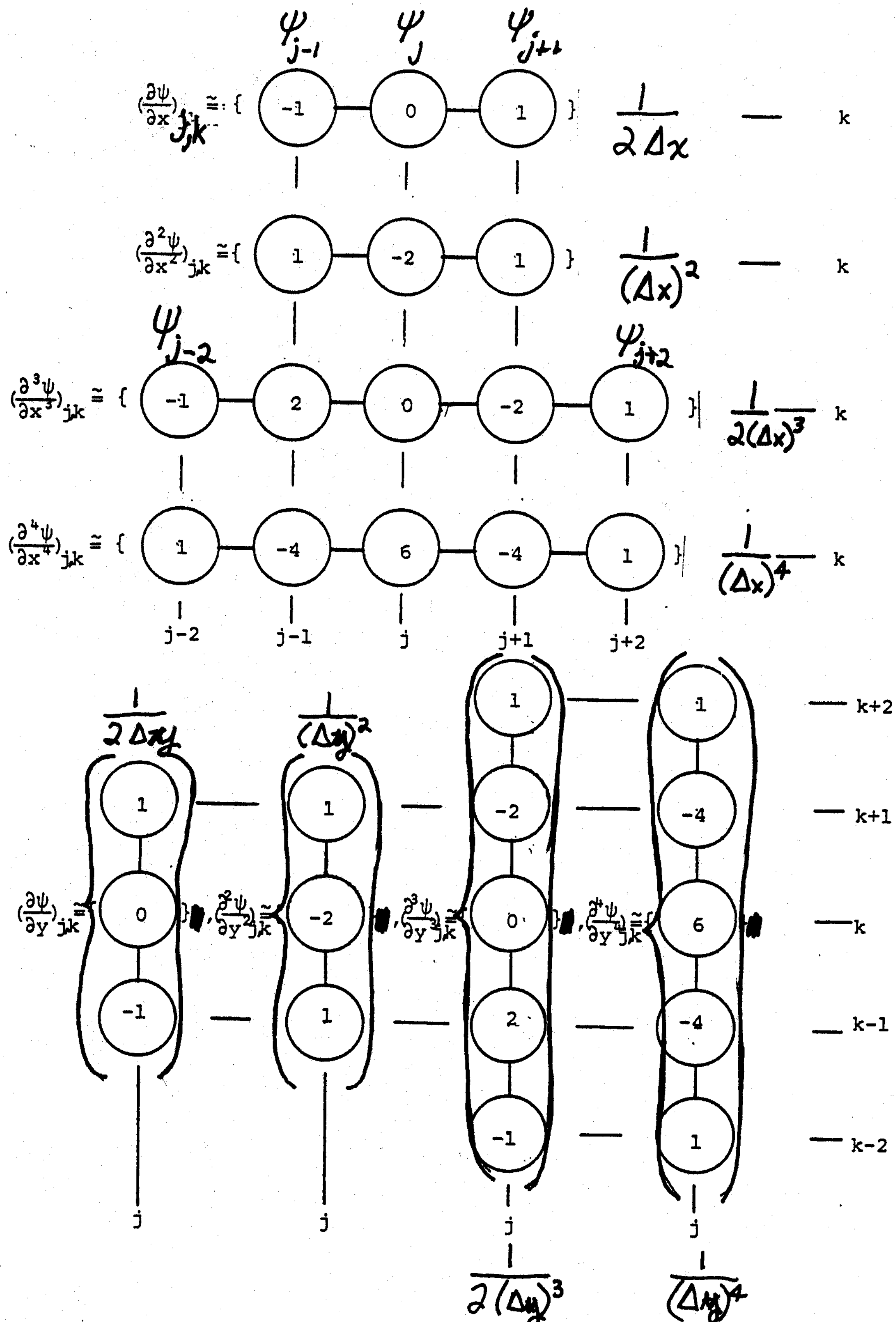


Figure 2.4 The finite difference operators for first few derivatives of $\psi=\psi(x,y)$ in x- and y-directions at point [j,k] with [redacted] $O(\Delta x)^2$ approximation