Math 2108

<u>Def</u>: $\forall x \in \mathbb{R}$, set

$$e^x := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

where, for example,

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4! \text{ etc.}$$

We saw that, $\forall x, y \in \mathbb{R}$

$$e^x \cdot e^y = e^{x+y}$$

We also saw that

$$f(x) = e^x \Rightarrow f'(x) = e^x = f(x), \ \forall x \in \mathbb{R}.$$

Now, $\forall x \geq 0$,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \geq 1 > 0$$

$$\therefore \quad e^{\text{positive}} = \text{Positive}$$

Also $\forall x > 0$

$$e^{-x} \cdot e^x = e^0 = 1$$

So

$$e^{-x} = \frac{1}{e^x}$$

When $x > 0$, $-x < 0$

$$\therefore \quad e^{\text{neg}} = e^{-x} = \frac{1}{e^x} = \frac{1}{e^{\text{pos}}} = \frac{1}{\text{pos}} = \text{Pos}.$$

$\therefore$ $e^{\text{pos}} = \text{pos}$, $e^0 = 1$, & $e^{\text{neg}} = \text{Pos}$.

$\therefore \quad e^x > 0 \quad \forall\, x \in \mathbb{R}$

i.e., the graph of

$$y = f(x) = e^x$$

is always above the x-axis.

Moreover

$$\lim_{x \to +\infty} e^x = \lim_{x \to +\infty} \left(1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$> \lim_{x \to +\infty} (1 + x) = +\infty.$$

Finally,

$$\lim_{x \to -\infty} e^x = \lim_{x \to +\infty} e^{-x} = \lim_{x \to +\infty} \left(\frac{1}{e^x}\right)$$

$$= \frac{1}{\lim_{x \to +\infty} (e^x)} = 0$$

Finally,

1. $f(x) = e^x > 0 \qquad \forall\, x \in \mathbb{R}$

$\Downarrow$

2. $f'(x) = e^x > 0 \qquad \forall\, x \in \mathbb{R} \Rightarrow f \nearrow$

$\Downarrow$

3. $f''(x) \;\; e^x > 0 \qquad \forall\, x \in \mathbb{R} \Rightarrow$ 

By ①, $f(x) = e^x > 0 \quad \forall x > 0$

so the graph of f is always above the x-axis.

By ②, f is continuous everywhere (since f differentiable $\Rightarrow$ f continuous)

so the graph of f is connected (i.e., in a single piece). Also

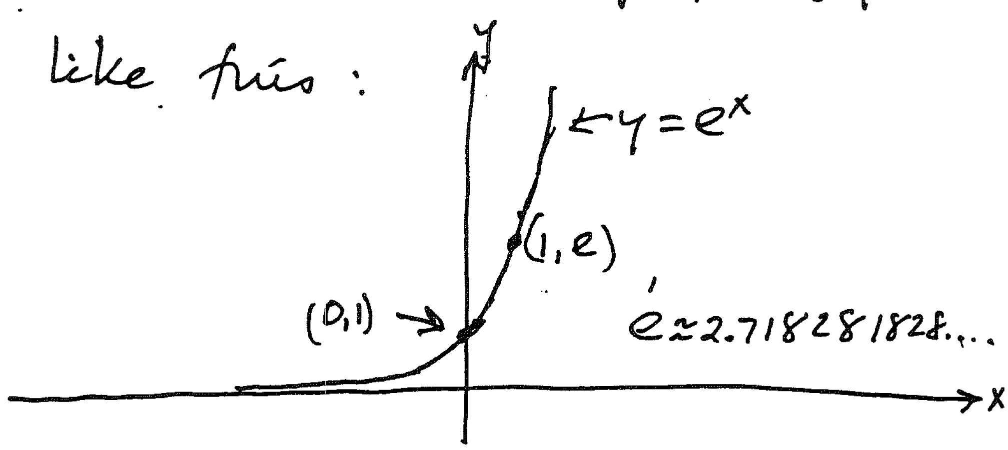
$f' > 0 \Rightarrow f \uparrow$, i.e., f is strictly increasing everywhere.

By ③, the graph of f is everywhere concave-up.

Finally, from the fact that

$\lim_{x \to +\infty} e^x = +\infty$ & $\lim_{x \to -\infty} e^x = 0$

we conclude that the graph of f looks like this:



Then $f(x) = e^x$ has

$$f: \mathbb{R} \to \mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}.$$

Since $f \uparrow$, f is 1-1. Also

Since f is continuous (everywhere)

& $\lim_{x \to +\infty} f(x) = +\infty$ & $\lim_{x \to -\infty} f(x) = 0$,

$$f: \mathbb{R} \longrightarrow \mathbb{R}^+$$

is onto as well as one-to-one.

Thus, $f: \mathbb{R} \to \mathbb{R}^+$ given by $f(x) = e^x$

has a unique inverse function

$$g: \mathbb{R}^+ \longrightarrow \mathbb{R}$$

<u>Def</u>: $g(x) = \ln(x) = \log_e(x)$

The natural logarithm of x

The logarithm base e of x.

Then for $f(x) = e^x > 0$ $(\forall x \in \mathbb{R})$

& $g(x) = \ln(x) \in \mathbb{R}$ $(\forall x > 0)$

we have that

⑤

$$x = g\circ f(x) = g[f(x)] = \log_e(e^x)$$
$$\qquad\qquad\qquad \overset{\shortparallel}{\ln(e^x)}$$

i.e.,

$$\boxed{\log_e(e^x) = x = \ln(e^x), \quad \forall x \in \mathbb{R}}$$

Q: How do you get the power out of an exponential function?

A. Hit it with a log.

Further

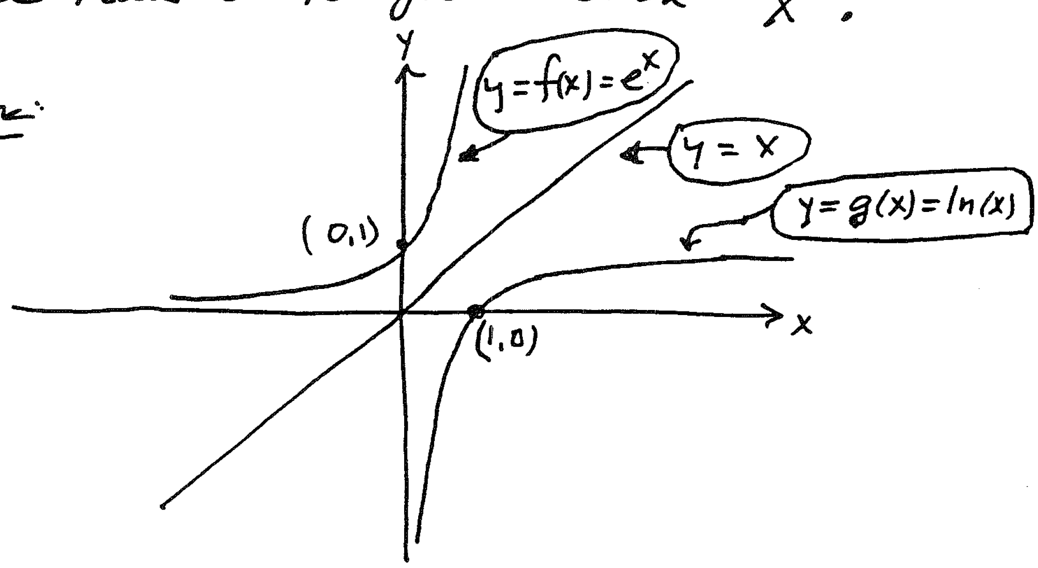
$$x = f\circ g(x) = f[g(x)] = e^{g(x)} = e^{\log_e x}$$
$$\qquad\qquad\qquad \overset{\shortparallel}{e^{\ln x}}$$

i.e.,

$$\boxed{e^{\log_e x} = x = e^{\ln x}, \quad \forall x > 0}$$

Thus, log base e of x is the power to which the base e must be raised to give back x.

Picture:



Claim: Using the facts that $e^{\ln x} = x$ & $\ln e^x = x$ we can prove that

$$\left.\begin{array}{c} e^x \cdot e^y = e^{x+y}, \\ \forall x, y \in \mathbb{R} \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \ln(x \cdot y) = \ln x + \ln y, \\ \forall x, y > 0 \end{array}\right.$$

Proof: $\forall x > 0$ & $y > 0$

$$x \cdot y = e^{\ln x} \cdot e^{\ln y} = e^{\ln x + \ln y}.$$

Hence

$$\ln(x \cdot y) = \ln\left[e^{\ln x + \ln y}\right] = \ln x + \ln y.$$

Conversely, we have the following

Claim: Using the identities $e^{\ln x} = x$ & $\ln e^x = x$ we can prove that

$$\left.\begin{array}{c} \ln(x \cdot y) = \ln x + \ln y, \\ \forall x, y > 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} e^x \cdot e^y = e^{x+y} \\ \forall x, y \in \mathbb{R} \end{array}\right.$$

Proof: All we are saying here is that if one just has that $f(x) = e^x$ & $g(x) = \ln x$ are inverse functions then the functional equation satisfied by g, namely, $g(x \cdot y) = g(x) + g(y)$ implies that satisfied by f, namely $f(x) \cdot f(y) = f(x+y)$, and conversely!

$\forall x, y \in \mathbb{R}$, $e^x, e^y > 0$, so

$$\ln(e^x \cdot e^y) = \ln(e^x) + \ln(e^y) = x + y.$$

Hence $e^{x+y} = e^{\ln(e^x \cdot e^y)} = e^x \cdot e^y, \quad \forall x, y \in \mathbb{R}.$

We now turn to the problem of defining a^x for any $a>0$ & any $x \in \mathbb{R}$.

Naturally, we wish to do this in such a way that $a^x = e^x$ when

$$a = e := e^x\Big|_{x=1} = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} + \frac{1^6}{6!} + \cdots$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5640} + \cdots$$

$$\approx 2.7 \ \underbrace{1828}_{\substack{\text{Andrew} \\ \text{Jackson} \\ \text{came to power in 1828}}} \ 1828 \ 45 \ 90 \ 45 \ 23 \ 53 \ 6 \ldots$$

$$\therefore e \approx 2.7\,(1828)(1828)$$

$$= 2.7\,(A.J.)\cdot(A.J.)$$

$$= 2.7\,(A.J.)^2 =$$ "two point seven Andrew Jackson squared"

Now, for any $a>0$

$$\ln(a^2) = \ln(a\cdot a) = \ln a + \ln a = 2\cdot\ln a,$$

$$\ln(a^3) = \ln(a\cdot a^2) = \ln a + \ln(a^2) = \ln a + 2\ln a$$

$$= 3\cdot\ln a.$$

In general, by induction on n, we find that for any positive integer n we have that $\ln(a^n) = n\cdot\ln a, \ \forall a>0.$

Now, from

(1) $\ln(a^n) = n \cdot \ln a$, $\forall$ positive integer n & any $a>0$

we should like, if possible, to have

(2) $\ln(a^x) = x \cdot \ln a$, $\forall x \in \mathbb{R}$ & any $a>0$.

For then (1) would be a special case of (2), or equivalently, (2) would be a generalization of (1), all in accord with "Hankel's Principle of the Permanence of Calculating Rules:

If we wish to extend a concept in Mathematics beyond its original definition, then among all the possible directions of this extension the one is to be chosen that will leave the calculating rules in tact as far as possible." – as quoted on page 27 of "Introduction to Mathematical Thinking" by Friedrich Waisman of New College, Oxford University.

Waismann goes on to write:

"This principle of permanence is not a statement whose validity may be questioned; instead it is, so to speak, a guiding principle for the formation of concepts."

To return to the matter at hand, this means that if we wish to give meaning to

$$y = a^x \quad \text{for } a>0 \ \& \ x \in \mathbb{R}$$

then we wish to do it in such a way that $y>0$ and

$$\ln y = \ln a^x = x \cdot \ln a$$

But it would then necessarily follow that

$$a^x = y = e^{\ln y} = e^{x \cdot \ln a}.$$

In short, the desire to give meaning to

$$y = a^x \quad \text{for arbitrary } a>0 \ \& \ x \in \mathbb{R}$$

in such a way that $y>0$ and $\ln(a^x) = x \cdot \ln a$ <u>forces us to define</u>

$$a^x := e^{x \cdot \ln a}.$$

Of course once we make this definition it then follows from the first boxed identity on page 5 that

$$\ln(a^x) = \ln\left(e^{x \cdot \ln a}\right) = x \cdot \ln a.$$

Isn't that GREAT?!?

The property we wanted forced us to make the definition that then guaranteed the desired property!

So if you ever have trouble recalling the definition of

$y = a^x$ for $a > 0$ & $x \in \mathbb{R}$

Simply recall that you also want

$$\ln y = \ln a^x = x \cdot \ln a.$$

That forces

$$a^x = y = e^{\ln y} = e^{x \cdot \ln a}$$

from which we have the required definition, namely, that

$$a^x := e^{x \cdot \ln a} \quad \text{for } a > 0 \text{ \& } x \in \mathbb{R}$$

Notice also that in the definition of

$$y = a^x := e^{x \cdot \ln a}$$

we must have $a > 0$ since the function

$$\ln : \mathbb{R}^+ \longrightarrow \mathbb{R}$$

has $\mathbb{R}^+$ as its domain!

Notice also that when $a = e$

$$a^x := e^{x \cdot \ln e} = e^{x \cdot 1} = e^x$$

(Since $\ln e = \ln e^1 = 1$) & this is as it should be!

Now for $z = b^y$ with $b > 0$ we require

$$\ln z = \ln b^y = y \cdot \ln b$$

& so

$$b^y = z = e^{\ln z} = e^{y \cdot \ln b}$$

Thus, for $b = a^x$, we have

$$(a^x)^y = b^y = e^{y \cdot \ln b} = e^{y \cdot \ln (a^x)}$$

$$= e^{y \cdot x \cdot \ln a} = e^{xy \ln a} = a^{x \cdot y}$$

thereby proving that $\forall\, a > 0$ & $\forall\, x, y \in \mathbb{R}$

$$(a^x)^y = a^{x \cdot y}.$$

Remarks:

(1) $$a^x \cdot a^y = a^{x+y}$$

is called The Additive Law of Exponents (ALOE) because we add the exponents

(2) $$(a^x)^y = a^{x \cdot y}$$

is called the Multiplicative Law of Exponents (MLOE) because we multiply the exponents.

Once again, let

$$f(x) = e^x \quad \& \quad g(x) = \ln x$$

Then $\quad x = f \circ g(x) = f[g(x)]$

Then $\dfrac{dx}{dx} = 1$ & by the Chain Rule

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

so

$$1 = \frac{dx}{dx} = (f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

$$= f[g(x)] \cdot g'(x) = x \cdot g'(x)$$

Since $f(x) = e^x \Rightarrow f'(x) = e^x = f(x) \ \forall x \in \mathbb{R} \Rightarrow f' = f : \mathbb{R} \to \mathbb{R}^+$

& therefore $f'[g(x)] = f[g(x)] = f \circ g(x) = id(x) = x$.

Then $1 = x \cdot g'(x) \Rightarrow g'(x) = \frac{1}{x}$, so $\dfrac{d \ln(x)}{dx} = \dfrac{1}{x}$.

Alternatively, from

$$f(x) = e^x$$

we get

$$f'(x) = e^x$$

Hence, for any differentiable function g of x, if we set

$$y = g(x)$$

we then find, by the Chain Rule that

$$\frac{d\, e^{g(x)}}{dx} = \frac{d\, e^y}{dx} = \frac{d\, e^y}{dy} \cdot \frac{dy}{dx} = e^y \cdot \frac{dy}{dx}$$

$$= e^{g(x)} \cdot g'(x),$$

i.e.,

$$\frac{d\, e^{g(x)}}{dx} = e^{g(x)} \cdot g'(x).$$

In particular, for $y = g(x) = \ln x$

$$e^{g(x)} = e^{\ln x} = x \quad \text{so}$$

$$1 = \frac{dx}{dx} = \frac{d\, e^{g(x)}}{dx} = e^{g(x)} \cdot g'(x) = x \cdot g'(x)$$

so

$$g'(x) = \frac{1}{x}, \quad \text{i.e.,} \quad \frac{d \ln x}{dx} = \frac{1}{x}.$$

Now let $z = h(x)$ with

h a differentiable function of x.

By the Chain Rule

$$\frac{d\ \ln[h(x)]}{dx} = \frac{d\ \ln z}{dx} = \frac{d\ \ln z}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{1}{z} \cdot \frac{dz}{dx} = \frac{1}{h(x)} \cdot h'(x)$$

i.e.,

$$\frac{d\ \ln[h(x)]}{dx} = \frac{1}{h(x)} \cdot h'(x)$$

$$= \frac{h'(x)}{h(x)}.$$

Next, since $e^{\log_e x} = x$ & $\log_e(e^x) = x$

we want $a^{\log_a x} = x$ & $\log_a a^x = x$.

So if $y = \log_a x$

we want $a^y = a^{\log_a x} = x$

This forces

$$y \cdot \ln a = \ln(a^y) = \ln x$$

or

$$\log_a x = y = \frac{\ln x}{\ln a} = \frac{\log_e x}{\log_e a}.$$

Next, if $y = a^{g(x)}$ for $a > 0$

$$\ln y = \ln\left[a^{g(x)}\right] = g(x) \cdot \ln a$$

so

$$a^{g(x)} = y = e^{\ln y} = e^{g(x) \cdot \ln a}$$

so

$$\frac{d\, a^{g(x)}}{dx} = e^{g(x) \cdot \ln a} \cdot g'(x) \cdot \ln a$$

$$= a^{g(x)} \cdot g'(x) \cdot \ln a$$

<u>Note:</u> If $a = e$, $\ln e = 1$ so this reduces to

$$\frac{d\, e^{g(x)}}{dx} = e^{g(x)} \cdot g'(x)$$

Next, if $y = [f(x)]^c$, $c \in \mathbb{R}$, $f(x) > 0$,

$$\ln y = \ln[f(x)]^c = c \cdot \ln f(x)$$

$$[f(x)]^c = y = e^{\ln y} = e^{c \cdot \ln f(x)}$$

so

$$\frac{d\,[f(x)]^c}{dx} = e^{c \cdot \ln f(x)} \cdot c \cdot \frac{1}{f(x)} \cdot f'(x)$$

$$= c \cdot \frac{[f(x)]^c}{[f(x)]^1} \cdot f'(x) = c \cdot [f(x)]^{c-1} \cdot f'(x)$$

<u>Note:</u> $c \cdot \frac{f^c}{f^1} \cdot f' = c \cdot f^c \cdot \frac{1}{f^1} \cdot f' = c \cdot f^c \cdot f^{-1} \cdot f'$

$= c \cdot f^{c+-1} \cdot f' = c \cdot f^{c-1} \cdot f'$

Putting all of this together we find that if

$$y = [f(x)]^{g(x)} \quad \text{for } f(x) > 0$$

then

$$\ln y = \ln [f^g] = g \cdot \ln f$$

so

$$f^g = y = e^{\ln y} = e^{g \cdot \ln f}.$$

Consequently,

$$(f^g)' = (e^{g \cdot \ln f})' = e^{g \cdot \ln f} \cdot \left[g \cdot \frac{1}{f} \cdot f' + \ln f \cdot g' \right]$$

$$= f^g \cdot \left[g \cdot \frac{1}{f} \cdot f' + \ln f \cdot g' \right]$$

$$= g \cdot f^g \cdot f^{-1} \cdot f' + f^g \cdot g' \cdot \ln f$$

$$= \underbrace{g \cdot f^{g-1} \cdot f'}_{①} + \underbrace{f^g \cdot g' \cdot \ln f}_{②}$$

Note: $\forall\, c \in \mathbb{R}$, $(f^c)' = c \cdot f^{c-1} \cdot f'$

while $\forall\, a \in \mathbb{R}$ with $a > 0$, $(a^g)' = a^g \cdot g' \cdot \ln a$

Hence $(f^g)'$ is the sum of 2 terms:

① is what we'd get if $g = c =$ constant for then $g' = c' = 0$; while

② is what we'd get if $f = a =$ a positive constant for then $f' = a' = 0$.

Finally, when $g = c$, the formula for $(f^g)'$ gives that for $(f^c)'$; while when $f = a$, the formula for $(f^g)'$ gives that for $(a^g)'$! ISN'T THAT GREAT ?!?