Math 2107

An Extra Sample Exam 1

1. Evaluate the following limit.
Use the symbols DNE (= Does Not Exist), $+\infty$, or $-\infty$ when applicable.
Place a box around your answer.

[60 points]
(10 pts each)

a. $\lim_{x \to -1} \dfrac{x^3 - x^2 + x + 2}{x - 1} =$

b. $\lim_{x \to 0} \sqrt[3]{x-8} - \sqrt[3]{x-1} =$

c. $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 7x + 10} =$

d. $\lim_{x \to 3} \dfrac{\sqrt{x-2} - 1}{x - 3}$

e. $\displaystyle \lim_{x \to 4^-} \frac{4-x}{|x-4|} =$

f. $\displaystyle \lim_{x \to 3^+} \frac{1}{3-x}$

g. $\displaystyle \lim_{x \to 0} \frac{|x|}{x}$

h. $\displaystyle \lim_{x \to \frac{\pi}{2}^+} \sec x =$

i. $\displaystyle \lim_{x \to 0} \frac{\tan(7x)}{\sin(3x)} =$

j. $\displaystyle \lim_{x \to 0} \frac{1-\cos x}{x^2} =$

2\. a. Find the vertical asymptotes to the graph of $y = f(x) = \dfrac{x^3}{x^2+x-6}$

[2 points]

b\. Find the slant asymptote to the graph of $y = f(x) = \dfrac{x^3}{x^2+x-6}$.

[6 points]

c. Find the horizontal asymptotes to the graph of $y = f(x) = \dfrac{-6x}{\sqrt{9x^2+4}}$.

[8 points]

3.a. Let $f(x) := \begin{cases} x^3+3 & , \text{ if } x<-2 \\ 0 & , \text{ if } x=-2 \\ 4-\frac{3}{2}x & , \text{ if } x>-2 \end{cases}$.

(i) Does $\lim_{x \to -2} f(x)$ exist?

Answer: ________________
(Yes or No)

[1 point]

Explain why or why not.

[2 points]

(ii) Is $f : \mathbb{R} \to \mathbb{R}$ continuous at $x=-2$?

Answer: ________________
(Yes or No)

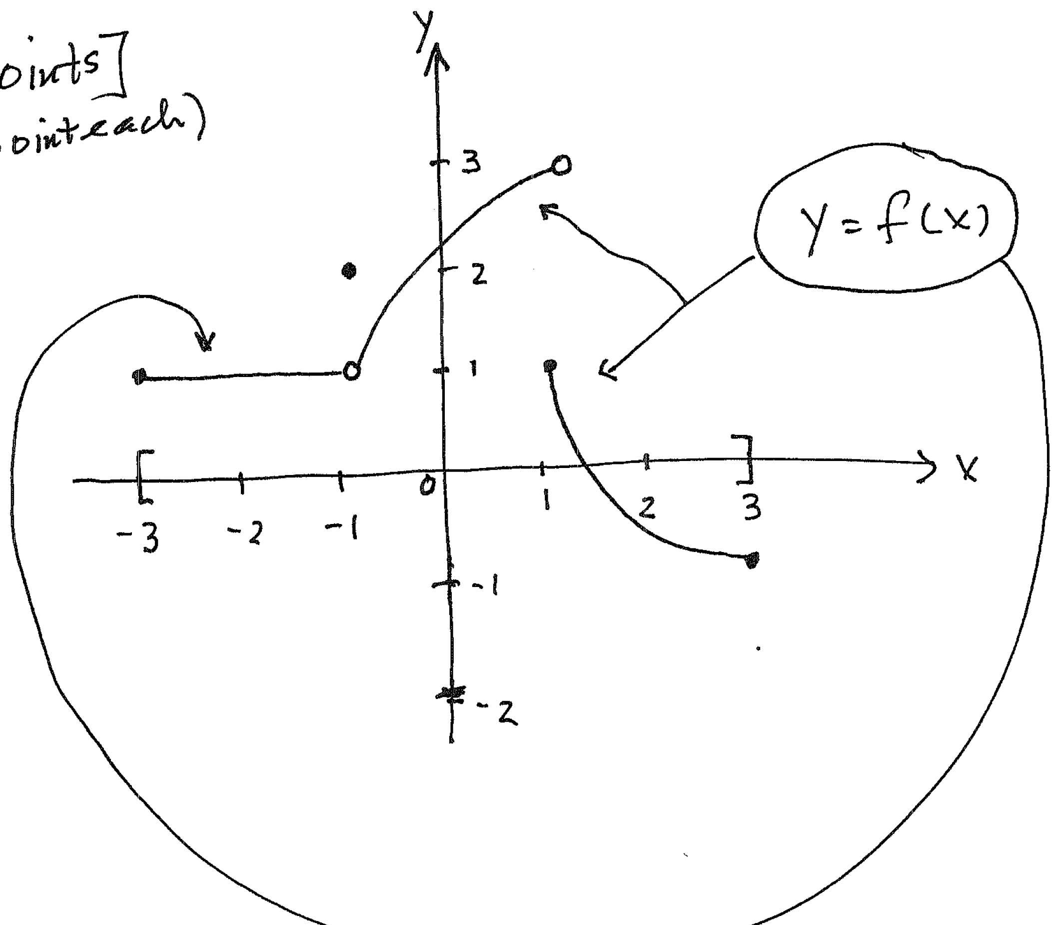
[1 point]

Explain why or why not.

[2 points]

3.b. With reference to the graph

[6 points]
(1 point each)



answer each of the following questions about the function $f : [-3,3] \to [-\frac{6}{7}, 3)$.

(i) $\displaystyle\lim_{x \to -1^-} f(x) =$ ______.

(ii) $\displaystyle\lim_{x \to -1^+} f(x) =$ ______.

(iii) $\displaystyle\lim_{x \to -1} f(x) =$ ______.

(iv) $f(-1) =$ ______.

(v) $\displaystyle\lim_{x \to 1^-} f(x) =$ ______.

(vi) List all the points at which $f : [-3,3] \to [-\frac{6}{7}, 3)$ is not continuous.

Answer: ______.

3.c. Let $f(x) := \begin{cases} x^2, & \text{if } x \le 3 \\ ax^2 + 2x - 15, & \text{if } x > 3. \end{cases}$

Determine the value of a so that f is continuous at $x = 3$.

Solution:

[6 points]

Answer: $a =$ ____

3.d. Evaluate $\lim\limits_{x \to -\infty} \left(x + \sqrt{x^2 - 6x}\right)$.

[6 points] Hint: when $x < 0$, $x = -\sqrt{x^2}$

Since $\sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x \le 0 \end{cases} \Rightarrow \sqrt{x^2} = -x$ for $x < 0$

$\Downarrow$

$x = -\sqrt{x^2}$ for $x < 0$.

Solution:

Math 2107

An Extra Sample Exam 1

1. Evaluate the following limit.
Use the symbols DNE (= Does Not Exist), $+\infty$, or $-\infty$ when applicable.
Place a box around your answer.

[60 points]
(10 pts each)

a. $$\lim_{x\to -1} \frac{x^3 - x^2 + x + 2}{x-1} = \frac{\lim_{x\to -1}(x^3 - x^2 + x + 2)}{\lim_{x\to -1}(x-1)} = \frac{(-1)^3 - (-1)^2 + (-1) + 2}{(-1) - 1}$$
$$= \frac{-1-1-1+2}{-2}$$
$$= \frac{-1}{-2} = \boxed{\frac{1}{2}}$$

b. $$\lim_{x\to 0} \sqrt[3]{x-8} - \sqrt[3]{x-1} = \sqrt[3]{0-8} - \sqrt[3]{0-1}$$
$$= \sqrt[3]{-8} - \sqrt[3]{-1} = -2 - (-1) = -2 + 1$$
$$= \boxed{-1}$$

c. $$\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 - 7x + 10} = \lim_{x\to 2} \frac{(x+3)(x-2)}{(x-5)(x-2)} = \lim_{x\to 2} \frac{x+3}{x-5} = \frac{2+3}{2-5}$$
$$= \frac{5}{-3} = \boxed{-\frac{5}{3}}$$

d. $$\lim_{x\to 3} \frac{\sqrt{x-2} - 1}{x-3} = \lim_{x\to 3} \frac{\sqrt{x-2} - 1}{x-3} \cdot \frac{\sqrt{x-2} + 1}{\sqrt{x-2} + 1}$$
$$= \lim_{x\to 3} \frac{\left(\sqrt{x-2}\right)^2 - 1^2}{x-3} \cdot \frac{1}{\sqrt{x-2} + 1}$$
$$= \lim_{x\to 3} \frac{x-2-1}{x-3} \cdot \frac{1}{\sqrt{x-2} + 1} = \lim_{x\to 3} \frac{x-3}{x-3} \cdot \frac{1}{\sqrt{x-2}+1}$$
$$= \lim_{x\to 3} \frac{1}{\sqrt{x-2} + 1} = \frac{1}{\sqrt{3-2} + 1} = \frac{1}{\sqrt{1}+1} = \frac{1}{1+1}$$
$$= \boxed{\frac{1}{2}}$$

e. $\lim\limits_{x\to 4^-} \dfrac{4-x}{|x-4|} = \lim\limits_{\substack{x\to 4\\ x<4}} \dfrac{4-x}{|x-4|}$

$|x-4| = \begin{cases} x-4, & \text{if } x-4 \geq 0 \\ -(x-4), & \text{if } x-4 \leq 0 \end{cases} = \begin{cases} x-4, & \text{if } x \geq 4 \\ 4-x, & \text{if } x \leq 4 \end{cases}$

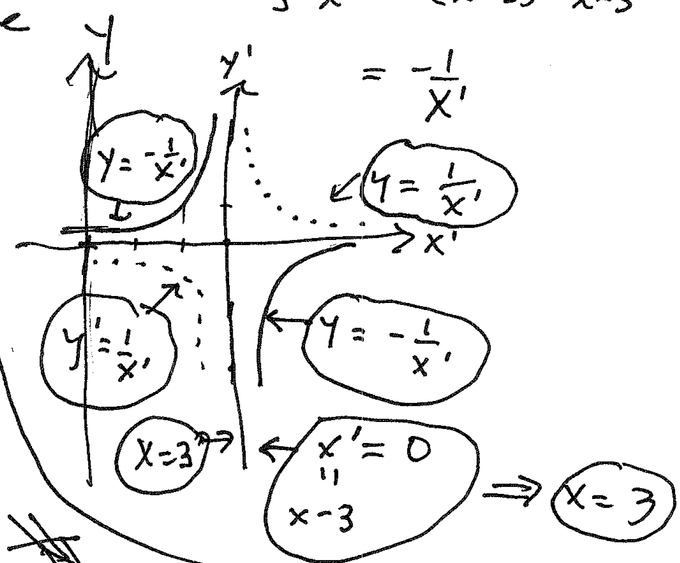
$= \lim\limits_{\substack{x\to 4\\ x<4}} \dfrac{4-x}{4-x} = \lim\limits_{\substack{x\to 4\\ x<4}} 1 = \boxed{1}$

f. $\lim\limits_{x\to 3^+} \dfrac{1}{3-x} = \dfrac{1}{0^-} = \boxed{-\infty}$

for $x>3$, $\underset{\substack{\uparrow\\ \text{smaller}}}{3} - \underset{\text{bigger}}{x} = \text{negative}$

$y' = y = \dfrac{1}{3-x} = \dfrac{1}{-(x-3)} = \dfrac{-1}{x-3}$

$= \dfrac{-1}{x'}$



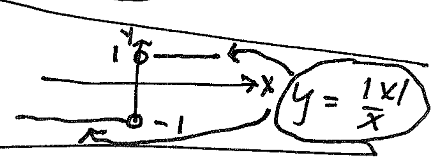
g. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ $\boxed{\text{DNE}}$

because

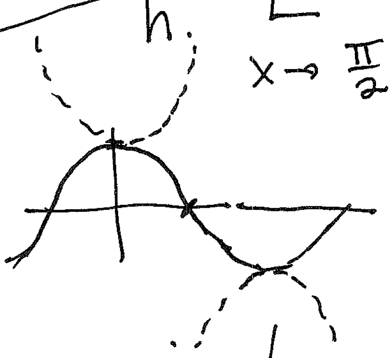
$\lim\limits_{x\to 0^+} \dfrac{|x|}{x} = \lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{|x|}{x} = \lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{x}{x} = \lim\limits_{\substack{x\to 0\\ x>0}} 1 = 1$

while

$\lim\limits_{x\to 0^-} \dfrac{|x|}{x} = \lim\limits_{\substack{x\to 0\\ x<0}} \dfrac{|x|}{x} = \lim\limits_{\substack{x\to 0\\ x<0}} \dfrac{-x}{x} = \lim\limits_{\substack{x\to 0\\ x<0}} -1 = -1$



h. $\lim\limits_{x\to \frac{\pi}{2}^+} \sec x = \lim\limits_{\substack{x\to \frac{\pi}{2}\\ x>\frac{\pi}{2}}} \dfrac{1}{\cos x} = \dfrac{1}{0^-} = \boxed{-\infty}$



i. $\lim\limits_{x\to 0} \dfrac{\tan(7x)}{\sin(3x)} = \lim\limits_{x\to 0} \dfrac{\frac{\tan(7x)}{7x}}{\frac{\sin(3x)}{3x}} \cdot \left(\dfrac{7x}{3x}\right)$

$= \lim\limits_{x\to 0} \dfrac{\frac{\tan(7x)}{7x}}{\frac{\sin(3x)}{3x}} \cdot \dfrac{7}{3} = \dfrac{1}{1} \cdot \dfrac{7}{3} = \boxed{\dfrac{7}{3}}$

j. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \dfrac{(\sin x)^2}{x^2} \cdot \dfrac{1}{1+\cos x}$

$\dfrac{1-c}{x^2} = \dfrac{1-c}{x^2} \cdot \dfrac{1+c}{1+c} = \dfrac{1^2-c^2}{x^2} \cdot \dfrac{1}{1+c} = \dfrac{s^2}{x^2} \cdot \dfrac{1}{1+c}$

$= \lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^2 \cdot \left(\dfrac{1}{1+\cos x}\right)$

$= 1 \cdot \left(\dfrac{1}{1+1}\right) = 1 \cdot \dfrac{1}{2} = \boxed{\dfrac{1}{2}}$

2. a. Find the vertical asymptotes to the graph of $y = f(x) = \dfrac{x^3}{x^2+x-6}$

[2 points] Solution:

$$\frac{x^3}{x^2+x-6} = \frac{x^3}{(x+3)\cdot(x-2)} \quad \& \quad (x+3)\cdot(x-2)=0$$

$$\Updownarrow$$

$$x=-3 \text{ or } x=+2$$

so the vertical asymptotes of $y=f(x)=\dfrac{x^3}{x^2+x-6}$ are $\boxed{x=-3}$ and $\boxed{x=+2}$.

b. Find the slant asymptote to the graph of $y = f(x) = \dfrac{x^3}{x^2+x-6}$.

[6 points] Solution:

$$\begin{array}{r|l} & x-1 \\ \hline x^2+x-6 & x^3+0\cdot x^2+0\cdot x^1+0\cdot x^0 \\ & x^3+x^2-6x \\ \hline & -x^2+6x+0 \\ & -x^2-x+6 \\ \hline & 7x-6 \end{array}$$

$$\begin{array}{r|l} & q \\ \hline d & n \\ \hline & r \end{array}$$

$$\Downarrow$$

$$\frac{n}{d} = q + \frac{r}{d}$$

$$\Updownarrow$$

$$n = q\cdot d + r$$

Hence

$$\frac{x^3}{x^2+x-6} = x-1 + \frac{7x-6}{x^2+x-6}$$

[i.e. $x^3 = (x-1)\cdot(x^2+x-6) + 7x-6$ [check this!]]

So $\displaystyle \lim_{x\to\pm\infty}\left[\frac{x^3}{x^2+x-6} - (x-1)\right] = \lim_{x\to\pm\infty}\frac{7x-6}{x^2+x-6}\cdot\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)$

$$= \lim_{x\to\pm\infty}\frac{\frac{7x-6}{x^2}}{\frac{x^2+x-6}{x^2}} = \lim_{x\to\pm\infty}\frac{\frac{7}{x}-\frac{6}{x^2}}{1+\frac{1}{x}-\frac{1}{x^2}} = \frac{0-0}{1} = 0$$

So $\boxed{y = x+1}$ is the slant asymptote of $y=f(x)=\dots$.

c. Find the horizontal asymptotes to the graph of $y = f(x) = \dfrac{-6x}{\sqrt{9x^2+4}}$.

[8 points] Solution:

$$\frac{-6x}{\sqrt{9x^2+4}} = \frac{-6x}{\sqrt{x^2\left(\frac{9x^2+4}{x^2}\right)}} = \frac{-6x}{\sqrt{x^2}\sqrt{9+\frac{4}{x^2}}}$$

$$= \frac{-6x}{|x|\cdot\sqrt{9+\frac{4}{x^2}}}$$

So, for $x>0$,

$$\frac{-6x}{\sqrt{9x^2+4}} = \frac{-6x}{x\sqrt{9+\frac{4}{x^2}}} = \frac{-6}{\sqrt{9+\frac{4}{x^2}}}$$

whereas, for $x<0$,

$$\frac{-6}{\sqrt{9x^2+4}} = \frac{-6x}{(-x)\cdot\sqrt{9+\frac{4}{x^2}}} = \frac{6}{\sqrt{9+\frac{4}{x^2}}}.$$

Consequently,

$$\underset{x\to+\infty}{L} \frac{-6x}{\sqrt{9x^2+4}} = \underset{x\to+\infty}{L} \frac{-6}{\sqrt{9+\frac{4}{x^2}}} = \frac{-6}{\sqrt{9+0}} = \frac{-6}{3} = -2$$

whereas

$$\underset{x\to-\infty}{L} \frac{-6x}{\sqrt{9x^2+4}} = \underset{x\to-\infty}{L} \frac{6}{\sqrt{9+\frac{4}{x^2}}} = \frac{6}{\sqrt{9+0}} = \frac{6}{3} = +2.$$

Hence the horizontal asymptotes to the graph of $y = f(x) = \dfrac{-6x}{\sqrt{9x^2+4}}$ are $\boxed{y=-2}$ and $\boxed{y=+2}$

3.a. Let $f(x) := \begin{cases} x^3+3 & \text{, if } x<-2 \\ 0 & \text{, if } x=-2 \\ 4-\frac{3}{2}x & \text{, if } x>-2 \end{cases}$.

(i) Does $\lim_{x \to -2} f(x)$ exist ?

Answer: Yes
(Yes or No)

[1 point]

Explain why or why not.

[2 points] $\lim_{x \to -2^-} f(x) = \lim_{\substack{x\to -2 \\ x<-2}} (x^2+3) = (-2)^2+3 = 4+3 = 7$

while $\lim_{x\to -2^+} f(x) = \lim_{\substack{x\to -2 \\ x>-2}} (4-\frac{3}{2}x) = 4-\frac{3}{2}(-2) = 4+3 = 7$

Because $\lim_{x\to -2^-} f(x) = 7 = \lim_{x\to -2^+} f(x)$, it follows that $\lim_{x\to -2} f(x) = 7$.

(ii) Is $f:\mathbb{R}\to\mathbb{R}$ continuous at $x=-2$?

Answer: No
(Yes or No)

[1 point]

Explain why or why not.

[2 points] $f(-2) = 0$ whereas $\lim_{x\to -2} f(x) = 7 \neq 0 = f(-2)$

But for f to be continuous at $a \in \mathbb{R}$ one must have

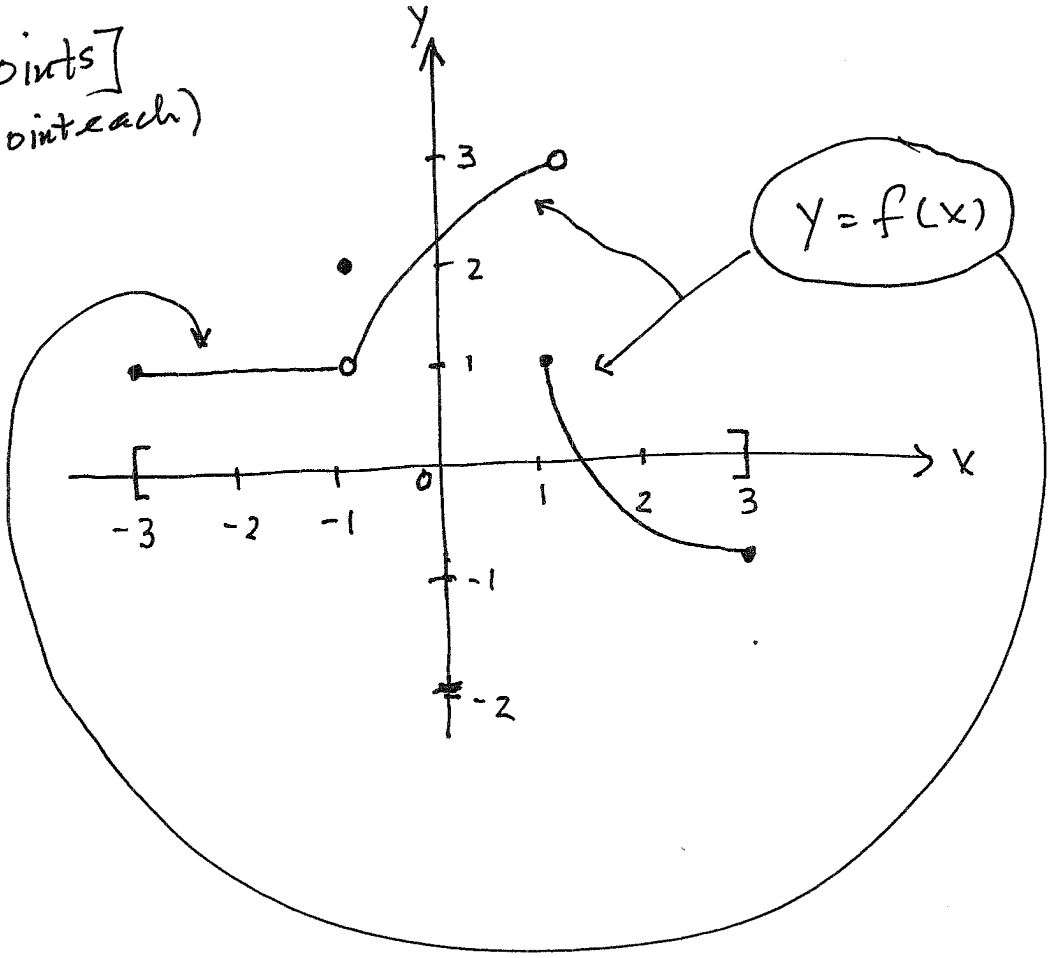
① f is defined at a (i.e., $f(a)$ exists)

② $\lim_{x\to a} f(x)$ exists,

& ③ $\lim_{x\to a} f(x) = f(a)$. Here ③ is violated!

3.b. With reference to the graph

[6 points]
(1 point each)



answer each of the following questions about the function $f : [-3, 3] \to [-\frac{6}{7}, 3)$.

(i) $\lim\limits_{x \to -1^-} f(x) = \underline{\quad 1 \quad}$, since $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{\substack{x \to -1 \\ x < -1}} f(x)$

(ii) $\lim\limits_{x \to -1^+} f(x) = \underline{\quad 1 \quad}$, since $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{\substack{x \to -1 \\ x > -1}} f(x)$

(iii) $\lim\limits_{x \to -1} f(x) = \underline{\quad 1 \quad}$, since $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^+} f(x)$

(iv) $f(-1) = \underline{\quad 2 \quad}$.

(v) $\lim\limits_{x \to 1^-} f(x) = \underline{\quad 3 \quad}$, since $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{\substack{x \to 1 \\ x < 1}} f(x)$

(vi) List all the points at which $f : [-3, 3] \to [-\frac{6}{7}, 3)$ is not continuous.

Answer: $x = -1$ & $x = +1$ are points of discontinuity of f.

3.c. Let $f(x) := \begin{cases} x^2, & \text{if } x \le 3 \\ ax^2 + 2x - 15, & \text{if } x > 3. \end{cases}$

Determine the value of a so that f is continuous at $x = 3$.

Solution: $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{\substack{x \to 3 \\ x < 3}} f(x) = \lim\limits_{\substack{x \to 3 \\ x < 3}} x^2 = 3^2 = 9$

[6 points] whereas

$$\lim_{x \to 3^+} f(x) = \lim_{\substack{x \to 3 \\ x > 3}} f(x) = \lim_{\substack{x \to 3 \\ x > 3}} ax^2 + 2x - 15 = a \cdot 3^2 + 2 \cdot 3 - 15$$
$$= 9a + 6 - 15$$
$$= 9a - (15 - 6)$$
$$= 9a - 9 = 9(a - 1)$$

Hence

$$9 = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 9(a-1)$$
$$\Downarrow$$
$$1 = \frac{9}{9} = a - 1 \Rightarrow a = 1 + 1 = 2$$

Answer: $a = 2$.

3.d. Evaluate $\lim\limits_{x \to -\infty} \left(x + \sqrt{x^2 - 6x}\right)$.

[6 points] Hint: when $x < 0$, $x = -\sqrt{x^2}$

Since $\sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x \le 0 \end{cases} \Rightarrow \sqrt{x^2} = -x$ for $x < 0$
$$\Downarrow$$
$x = -\sqrt{x^2}$ for $x < 0$.

Solution:

$$\lim_{x \to -\infty} \left(x + \sqrt{x^2 - 6x}\right) = \lim_{x \to -\infty} \frac{x + \sqrt{x^2 - 6x}}{1} \cdot \frac{x - \sqrt{x^2 - 6x}}{x - \sqrt{x^2 - 6x}}$$
$$= \lim_{x \to -\infty} \frac{x^2 - \left(\sqrt{x^2 - 6x}\right)^2}{x - \sqrt{x^2 - 6x}} = \lim_{x \to -\infty} \frac{x^2 - (x^2 - 6x)}{x - \sqrt{x^2 - 6x}}$$
$$= \lim_{x \to -\infty} \frac{x^2 - x^2 + 6x}{x - \sqrt{x^2 - 6x}} = \lim_{x \to -\infty} \frac{6x}{x - \sqrt{x^2 - 6x}}$$

3.d. (Continued)

Hence

$$\lim_{x \to -\infty} \left(x + \sqrt{x^2 - 6x} \right) = \lim_{x \to -\infty} \frac{6x}{x - \sqrt{x^2 - 6x}}$$

$$= \lim_{x \to -\infty} \frac{6x}{x - \sqrt{x^2 \left(\frac{x^2 - 6x}{x^2} \right)}}$$

$$= \lim_{x \to -\infty} \frac{6x}{x - \sqrt{x^2} \sqrt{1 - \frac{6}{x}}}$$

$$= \lim_{x \to -\infty} \frac{6x}{x - |x| \sqrt{1 - \frac{6}{x}}}$$

$$= \lim_{x \to -\infty} \frac{6x}{x - (-x) \sqrt{1 - \frac{6}{x}}}$$

$$= \lim_{x \to -\infty} \frac{6x}{x + x \sqrt{1 - \frac{6}{x}}}$$

$$= \lim_{x \to -\infty} \frac{x \cdot 6}{x \left(1 + \sqrt{1 - \frac{6}{x}} \right)}$$

$$= \lim_{x \to -\infty} \frac{6}{1 + \sqrt{1 - \frac{6}{x}}} = \frac{6}{1 + \sqrt{1 - 0^-}} = \frac{6}{1+1} = \frac{6}{2} = \boxed{3}$$