Math 2107

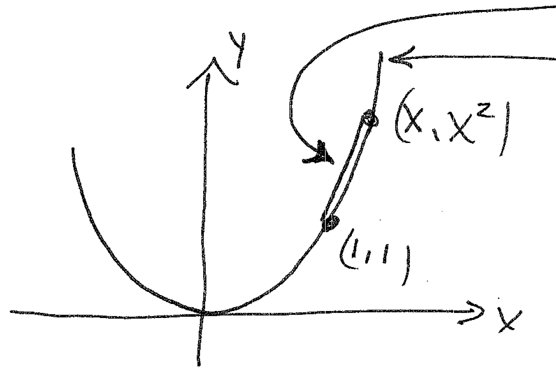
Introduction to Limits:

Consider the problem of finding the slope of the straight line tangent to the graph of $y = f(x) = x^2$ at the point $(1, f(1)) = (1,1)$.

By definition, this slope is equal to the limit as x approaches 1 of the difference quotient

$$Q(x) := \frac{x^2-1}{x-1}$$ = the slope of the secant line through $(1,1)$ & (x, x^2)

Picture:



$\forall$ = for all or for each or for every

Here, the difference quotient

$$Q(x) := \frac{x^2-1}{x-1}$$

is defined $\forall$ real $x \neq 1$. It is not defined if $x = 1$ for then $(x, x^2) = (1,1)$ & we do not have 2 distinct points & hence we don't have a line joining (x, x^2) & $(1,1)$. In addition if $x = 1$, then $x - 1 = 0$ & $x^2 - 1 = 0$ as well.

$P := Q$ means that Q is known & that P is defined to be equal to Q

i.e., $P := Q$ by definition: $P \overset{\text{def}}{=\!=} Q$

On the other hand, if $x \neq 1$, and - in fact - $\forall$ real $x \neq 1$, $Q(x)$ is defined & moreover

$\forall x \neq 1$,

$$Q(x) := \frac{x^2-1}{x-1} = \frac{(x-1)\cdot(x+1)}{x-1} = \frac{x-1}{x-1}\cdot(x+1)$$

$$= 1\cdot(x+1) = x+1 .$$

Thus, $\forall$ value of x for which $Q(x)$ is defined,

$$Q(x) = x+1 .$$

Consider, now, the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) := x+1 .$$

Here: $\mathbb{R}$ = the set of all real numbers

Then $\forall x \neq 1$, $Q(x) = g(x)$.

On the other hand as functions

$Q \neq g$, for

$$Q : \mathbb{R} - \{1\} \longrightarrow \mathbb{R}$$

Here, $\mathbb{R} - \{1\} = \{x \in \mathbb{R} \mid x \neq 1\}$ = the set of all real numbers x except $x = 1$.

while

$$g : \mathbb{R} \to \mathbb{R}$$

Recall, that a function consists of 3 things:

① a set X,

② a set Y,

& ③ a rule f which assigns to each element x of X exactly one element y of Y.

If $y \in \mathbb{Y}$ gets assigned to $x \in \mathbb{X}$ by f we denote this fact by writing

$$y = f(x)$$

& we call y the image under f of x

& we say

"y equals f of x"

or "y equals f at x"

or "y is the image of x under f."

We summarize the 3 pieces of data comprising a function by the symbol

$$f : \mathbb{X} \rightarrow \mathbb{Y}$$

which we read as "f colon $\mathbb{X}$ arrow $\mathbb{Y}$"

or "f from $\mathbb{X}$ to $\mathbb{Y}$"

or "the function f from $\mathbb{X}$ to $\mathbb{Y}$".

Notice that I distinguish between

lower case x and upper case $\mathbb{X}$

& lower case y and upper case $\mathbb{Y}$

by the horizontal bars at the head & feet of the symbols.

Note also that if

$\in$ = is an element of

$$f : X \to Y$$

is a function and $x \in X$

(i.e., lower case x is an element of the set X)

and $y = f(x)$, then we sometimes

denote the fact that x gets mapped

to y under f by writing

$$x \stackrel{f}{\longmapsto} y \quad \text{or} \quad x \mapsto f(x)$$

or more simply by

$$x \mapsto y$$

Here, we distinguish between the

<u>function arrow</u> "$\to$" as in

$f : X \to Y$ and the assignment

arrow "$\mapsto$" (having a

vertical bar at (i.e., perpendicular to)

the tail end of the arrow $\to$)

as in $x \mapsto y$.

If $f : X \to Y$ is a function then

the set X is called the <u>domain</u> of the

function $f : X \to Y$; the set Y is called

the <u>codomain</u> of the function $f : X \to Y$

& f is called the <u>rule</u> of the function $f : X \to Y$.

The set

$$f(X) := \{ f(x) \mid x \in X \}$$

is called the <u>image</u> of the function $f: X \to Y$. Some authors call $f(X)$ the <u>range</u> of $f: X \to Y$ while others call Y the <u>range</u> of $f: X \to Y$ so for this reason <u>we choose to avoid the word range altogether</u>. If $f: X \to Y$ is a function, then some authors call X the <u>source</u> of the function $f: X \to Y$ & Y the <u>Target</u> of the function $f: X \to Y$.

If $g: A \to B$ & $f: X \to Y$ are functions, then they are equal if & only if

$$A = X, \quad B = Y$$

& $g(x) = f(x) \quad \forall\, x \in X = A$.

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i.e., since a function consists of 3 things 2 functions are equal if & only if all 3 pieces of the data comprising them are the same!

Probably till now in your mathematical career you have been content to identify a function with its rule but a moments reflection shows that this is a fairly sloppy way of proceeding. Indeed, you are probably familiar with representing a function (from $\mathbb{R}$ to $\mathbb{R}$) by its graph as follows:

If $f : \mathbb{R} \to \mathbb{R}$ is a function then the graph of f, denoted Graph(f), is defined by

$$\text{Graph}(f) := \{(x, y) \mid y = f(x) \ \& \ x \in \mathbb{R}\}$$

$$= \{(x, f(x)) \mid x \in \mathbb{R}\}$$

More generally, if $X \subseteq \mathbb{R}$ & $Y \subseteq \mathbb{R}$ (i.e., if X & Y are subsets of the set $\mathbb{R}$ of all real numbers) and if $f : X \to Y$ is a function then $\text{graph}(f) := \{(x, f(x)) \mid x \in X\}$.

Recall next that if a & b are real numbers (i.e., if $a \in \mathbb{R}$ & $b \in \mathbb{R}$) & if a is less than b (i.e., if $a < b$), then we define the closed interval from a to b (including end points) by the symbol

$$[a,b] := \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

The open interval from a to b is defined by the symbol

$$(a,b) := \{ x \in \mathbb{R} \mid a < x < b \}$$

Some intervals are neither open nor closed; they are ajar:

$$(a,b] := \{ x \in \mathbb{R} \mid a < x \leq b \}$$

= the interval open at a & closed at b

while

$$[a,b) := \{ x \in \mathbb{R} \mid a \leq x < b \}$$

= the interval closed at a & open at b.

Note: The context should make it clear whether for real numbers a & b (a,b) = the open interval from a to b or whether (a,b) = the point with x-coordinate a & y-coordinate b!

Now consider the functions

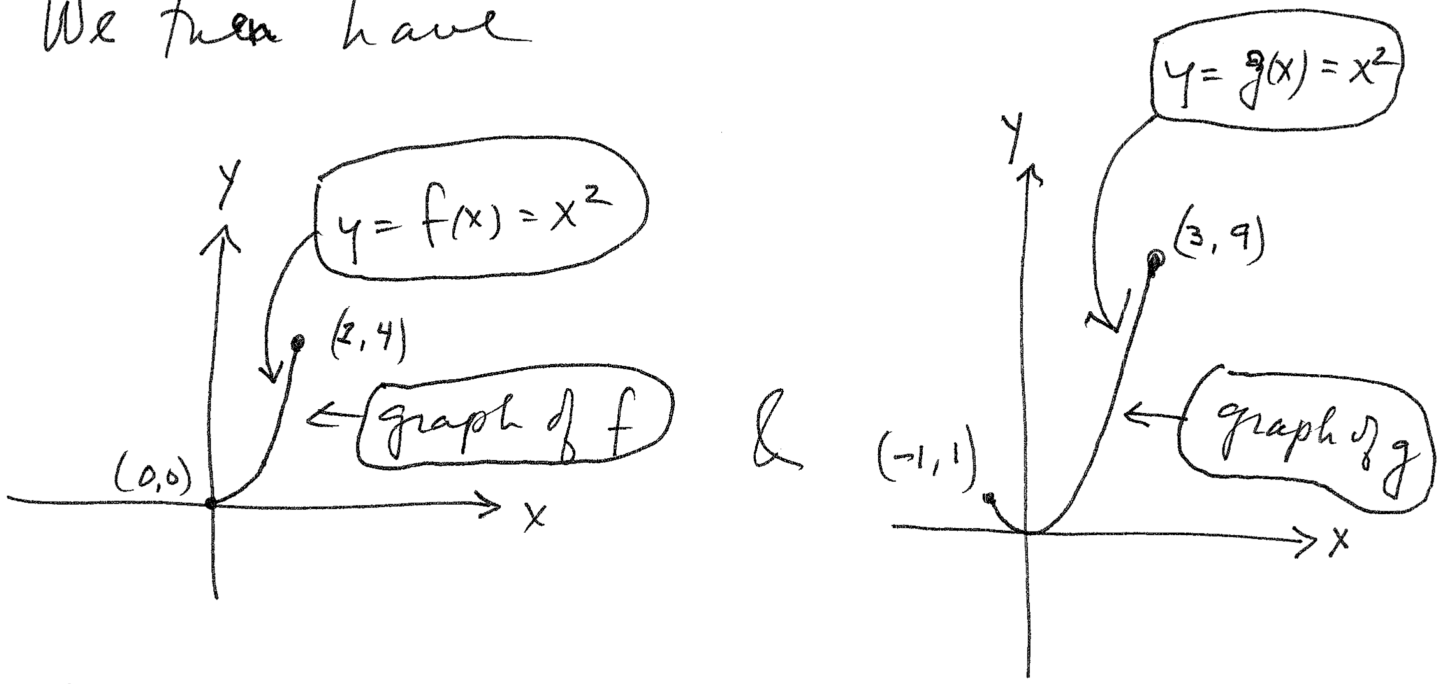
$$f : [0,2] \to [0,4]$$

&

$$g : [-1,3] \to \mathbb{R}$$

defined by $f(x) = x^2$ & $g(x) = x^2$

We then have



Clearly even though $f : [0,2] \to [0,4]$ & $g : [-1,3] \to \mathbb{R}$ both have the <u>same</u> <u>rules</u>, they are distinct as functions as one can readily appreciate by seeing how different their graphs are! In calculus & higher level courses we'll have to pay attention to when functions are <u>one-to-one</u> or <u>onto</u> or both or neither or one but not the other & these notions are only meaningful or rather can only be defined by considering a function as consisting of the 3 parts as above..

Back to the salt mines:

We were looking at

$$Q: \mathbb{R} - \{1\} \to \mathbb{R}$$

&

$$g: \mathbb{R} \to \mathbb{R}$$

defined by

$$Q(x) := \frac{x^2-1}{x-1} \quad \& \quad g(x) := x+1$$

Now consider $G: \mathbb{R} - \{1\} \to \mathbb{R}$ defined by $G(x) := x+1$.

As we know $\forall \; x \in \mathbb{R} - \{1\} := \{x \in \mathbb{R} \mid x \neq 1\}$

$$Q(x) = G(x).$$

Now $G: \mathbb{R} - \{1\} \to \mathbb{R}$ & $g: \mathbb{R} \to \mathbb{R}$ both have the same rule, namely,

$$x \longmapsto x+1.$$

Yet $G \neq g$ since their domains differ!

Specifically,

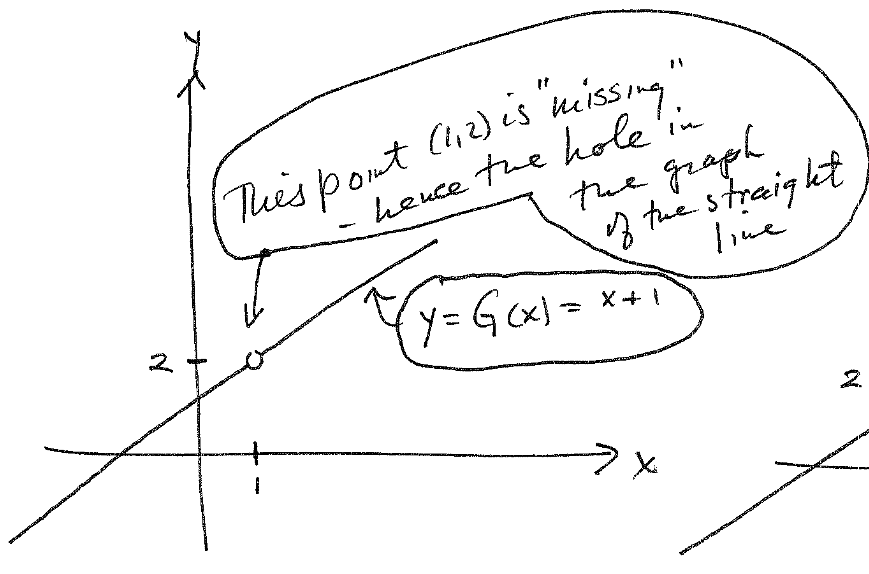
$$\text{Domain}(G) = \mathbb{R} - \{1\}$$

while

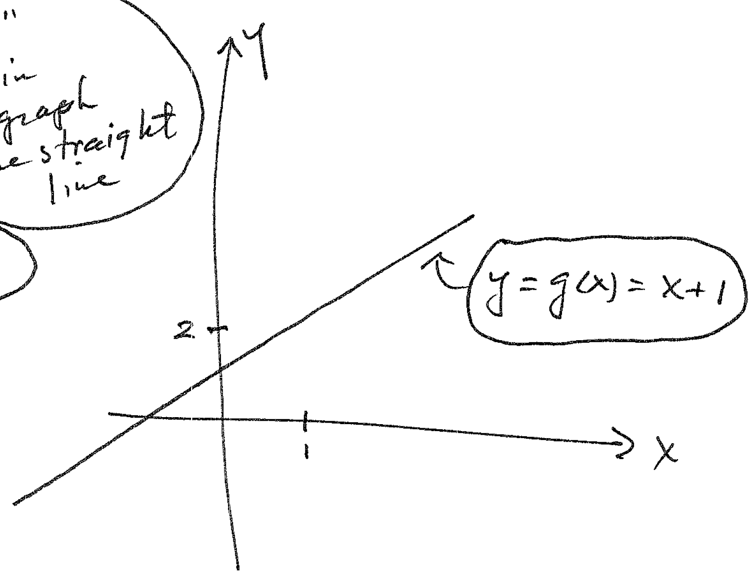
$$\text{Domain}(g) = \mathbb{R}.$$

Graphically, also, it is clear that $G \neq g$ since

$$\text{Graph}(G) \subseteq \text{Graph}(g).$$



Graph$(G : \mathbb{R} - \{1\} \to \mathbb{R})$



Graph$(g : \mathbb{R} \to \mathbb{R})$

Imagine now that we have a bug crawling along the straight line (that is part of the graph of G) towards the hole whose co-ordinates are $(1,2)$. As the x-value of the point $(x, G(x))$ on the graph of G gets closer & closer to 1 the y-value of $(x, G(x))$ will get closer & closer to 2.

We summarize this by saying that "as x approaches 1, $G(x)$ approaches 2" or by saying that "the limit of $G(x)$ as x approaches 1 is 2." In symbols

$$\lim_{x \to 1} G(x) = 2.$$

If some wise-ass comes by & suggests that as x approaches 1 $G(x)$ also approaches 2.001 how can we deal with him? How can we be sure that

$$\lim_{x \to 1} G(x) \neq 2.001?$$

Well, the idea is that we can make $G(x)$ as close to 2 as we like if only we take x sufficiently close to 1. In other words, we have a game here. Challenged with any error bound $\varepsilon > 0$ we must come up with a deviation bound $\delta > 0$ so that whenever x is within δ of 1 (but distinct from 1), then $G(x)$ will be within ε of 2.

δ = the lower case Greek letter "delta"
ε = " " " " "epsilon"

Let's think of this in another way. We talked earlier of functions as being represented by arrows & noted that some authors call the domain of a function its source & the codomain of a function its target. So this interpretation lends itself to thinking of a function as analogous to a bow & arrow together with the target. So suppose I claim to be a good shot. You challenge me: "Okay big boy, put your money where your mouth is. Let's see how good you are. Can you even hit the target?" Well hopefully I can do at least that well. But to prove I'm a good shot I'll have to get as close to the bull's eye as possible.

And if I'm really great, perhaps I can even split one arrow through the center of the bull's eye by a 2nd arrow! At any rate - I guess you get the idea. To prove I'm a good shot, I must show I can get as close to the center of the target as I'm challanged to do. And with functions & limits, it is precisely the same. If some one challenges ~~us~~ to get real close to the center of the target - say within ε of the center, then we'll have to take real close aim & allow the tail end of the arrow - by the string of the bow - to vary by just the slightest bit - say δ - so that once the arrow flies, it will land within the concentric circle of radius ε from the target's center. Do you see this in your mind's eye?

Prior to playing the "epsilon-delta" game with actual numbers, let's get a feel for it by looking at some pictures. By the way, this is a good & legitimate thing to do for it lends insight & thus helps you to get a feeling in your bones - in your finger-tips - for what is going on.

Another comment:

ε = the lower case greek letter
epsilon.

while
δ = the lower case greek letter
delta.

(The upper case greek letter delta = Δ.)

We were using the mnemonic device (= memory aide) of thinking of

"epsilon" as standing for "error-bound"

and

"delta" as standing for "deviation bound"

Note that both "epsilon" & "error-bound" start with the letter "e" (& ε=epsilon is the greek counterpart of "e") while both "delta" & "deviation-bound" start with the letter "d" (& δ=delta is the greek counterpart of "d"). So let's now review the definition of what it means to say that

$$\underset{x \to 1}{\text{Limit}}\ G(x) = 2.$$

where $G : \mathbb{R} - \{1\} \to \mathbb{R}$ is defined by

$$G(x) = x + 1.$$

Notice first of all that G is <u>not</u> defined at 1, so $G(1)$ does not make sense — even though for the function

$$g : \mathbb{R} \to \mathbb{R}$$

given by

$$g(x) = x + 1$$

$g(1)$ makes sense & $g(1) = 1 + 1 = 2$.

The point is that G is not defined at 1 & yet in this case the

$$\underset{x \to 1}{\text{Limit}}\ G(x)$$

exists. Notice next that at least in this case $G(x)$ is defined $\forall x \neq 1$.

& in particular $G(x)$ is defined $\forall\, x$ as close to (i.e., near to) 1 as desired. Later we shall remark that for an arbitrary function $f: X \to Y$ where X & Y are subsets of the set $\mathbb{R}$ of all real numbers, the symbol

$$\lim_{x \to a} f(x) = L$$

will be defined whenever a has points in the domain of f arbitrarily close to a, i.e., when

a = a <u>cluster point</u> or
an <u>accumulation point</u>
of the domain of f

By definition this means that for any positive real number r the open interval about a (i.e. centered at a) of radius r contains at least 1 point of the domain of f distinct from a (which is not required to be in the domain of f - i.e., f is not required to be defined at a).

All we are saying is that for the symbol

$$\underset{x \to a}{\text{Limit}}\ f(x) = L$$

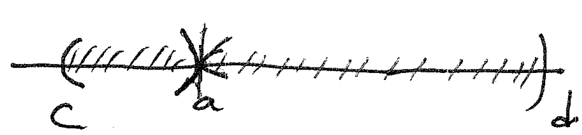
to be defined, we simply require that f be defined at points as close to a as required, but that f need not be defined at a. <u>Note</u>: This is just a "nicety" for you to file away somewhere in the back of your mind for future reference. As far as this course is concerned, we'll finese this point (i.e., simplify the subtleties) by requiring instead that f just be defined on an open interval containing a as an inside (interior) point but that f possibly is not defined at a. This means that there are real numbers c & d with

$c < a < d$ such that

$$(c,a) \cup (a,d) \subseteq \text{domain } f$$

$A \cup B =$ A union B $= \{x \mid x \in A \text{ or } x \in B\}$

<u>Picture</u>:



Certainly under this assumption, a will be an accumulation point of the domain of f.

Back to the task at hand:

We have the function

$$G : \mathbb{R} - \{1\} \to \mathbb{R}$$

defined by

$$G(x) = x + 1$$

& we wish to show that

$$\lim_{x \to 1} G(x) = 2.$$

By definition, this means:

① Challenged with any <u>error-bound</u>

$\varepsilon > 0$

② There exists a <u>deviation bound</u>

$\delta > 0$

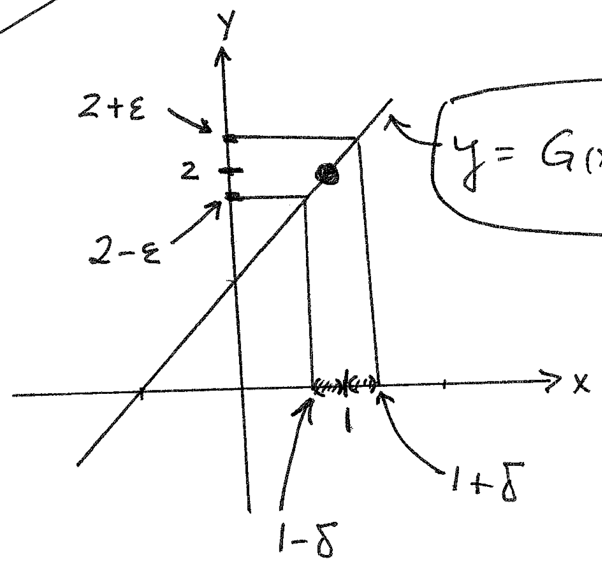
so that

③ whenever the distance between x and 1 is positive but less than δ

④ then the distance between $G(x)$ & 2 is less than ε.

Picture: To prove that $\lim_{x \to 1} G(x) = 2$

$\boxed{\text{Step 1}}$: We're challenged with $\varepsilon > 0$



$y = G(x) = x+1$, if $x \neq 1$

$\boxed{\text{Step 2}}$ We must come up with $\delta > 0$

so that

$\boxed{\text{Step 3}}$ whenever x is on the x-axis and either

$1-\delta < x < 1$ or $1 < x < 1+\delta$

$\boxed{\text{Step 4}}$ Then $G(x)$ = the image of x under G

is a point on the y-axis satisfying

$$2-\varepsilon < G(x) < 2+\varepsilon$$

The Inside Scoop: As you can readily see the easiest way to find a $\delta > 0$ corresponding to the $\varepsilon > 0$ one is challenged with is to draw horizontal lines through $2+\varepsilon$ & $2-\varepsilon$ & see where the intersect the graph of $y = G(x) = x+1$. Then draw the vertical lines through those points & see where they intersect the x-axis.

In the present case, since the function G is just a shift of the identity function $I : \mathbb{R} \to \mathbb{R}$ given by the rule

$$I(x) := x \quad , \forall x \in \mathbb{R},$$

an interval along the x-axis gets mapped by G to an interval of the same length along the y-axis.
Hence corresponding to an interval of radius ε centered at 2 (on the y-axis) we'll get an interval of radius ε centered at 1 on the x-axis; so given $\varepsilon > 0$ we can take $\delta = \varepsilon$ as we'll see momentarily. On the other hand if the function

$$f : \mathbb{R} - \{a\} \longrightarrow \mathbb{R}$$

is a straight line of slope m (with say $m > 1$) then an interval of radius $\delta > 0$ on the x-axis will get stretched to an interval of radius $m \cdot \delta$ on the y-axis

$\forall\, m \in \mathbb{R}$, $|m| := \begin{cases} m, & \text{if } m \geq 0 \\ -m, & \text{if } m \leq 0 \end{cases}$

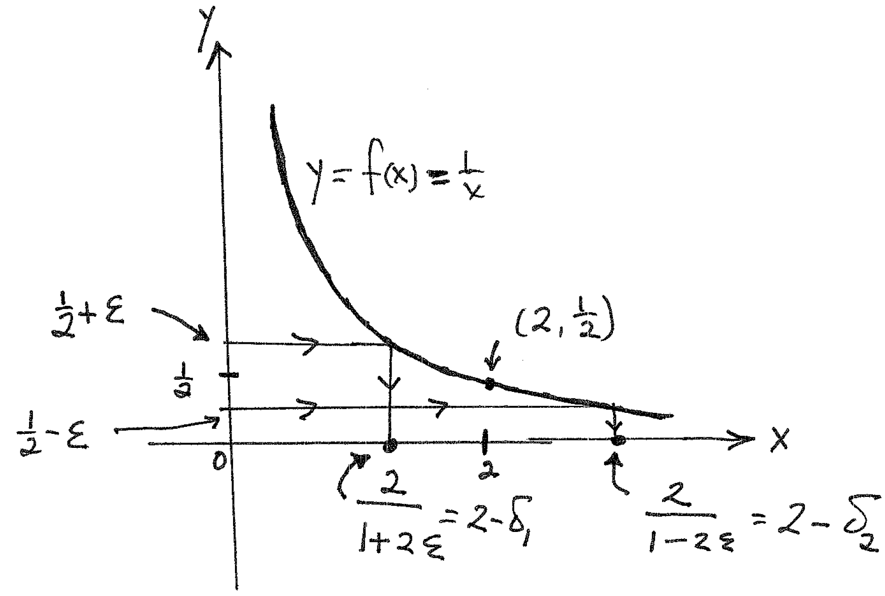
i.e., $|m|$ = the absolute value of m = the non-negative distance from m to 0.

If $0 < m < 1$, then an interval of length $\delta > 0$ on the x-axis will be shrunk by a factor of m on the y-axis. [Try $m = 2$ or $m = 3$ versus $m = \frac{1}{2}$ or $m = \frac{1}{3}$ to see what happens!] Of course if $m < 0$ then the stretching or shrinking factor is just $|m|$ since we just have a line with negative rather than positive slope.

For a more complicated picture leading at 1st blush to a non-symmetric picture consider the case of $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ & look at

$$\lim_{x \to 2} f(x) = \frac{1}{2}$$

Here the picture looks as follows:



As you can see, the corresponding interval on the x-axis is not having $x=2$ as its mid-point. In the above case

$$2-\delta_1 < x < 2-\delta_2 \implies \frac{1}{2}-\varepsilon < f(x) < \frac{1}{2}+\varepsilon$$

So here we just pick

$$\delta = \text{minimum of } \delta_1 \ \& \ \delta_2$$
$$= \min(\delta_1, \delta_2)$$

Then

$$2-\delta < x < 2 \ \underline{\text{or}} \ 2 < x < 2+\delta \implies \frac{1}{2}-\varepsilon < f(x) < \frac{1}{2}+\varepsilon$$

<u>Moral</u>: If any particular $\delta > 0$ "works" so does any smaller positive delta.

Practical Applications

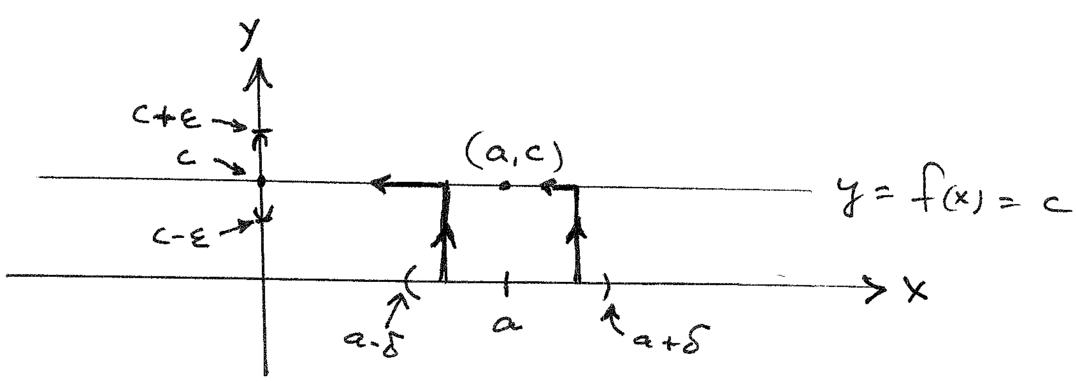
(1) Let's prove that if $f: \mathbb{R} \to \mathbb{R}$ is defined $\forall\, x \in \mathbb{R}$ via

$$f(x) = c = \text{some constant}$$

then $\forall\, a \in \mathbb{R}$

$$\lim_{x \to a} f(x) = \lim_{x \to a} c = c.$$

Picture: (For the case where $c > 0$ & $a > 0$).



given $\varepsilon > 0$ we may take $\delta > 0$ to be any positive real number for

$$0 < |x-a| < \delta \Rightarrow |f(x) - c| = |c - c| = 0 < \varepsilon$$

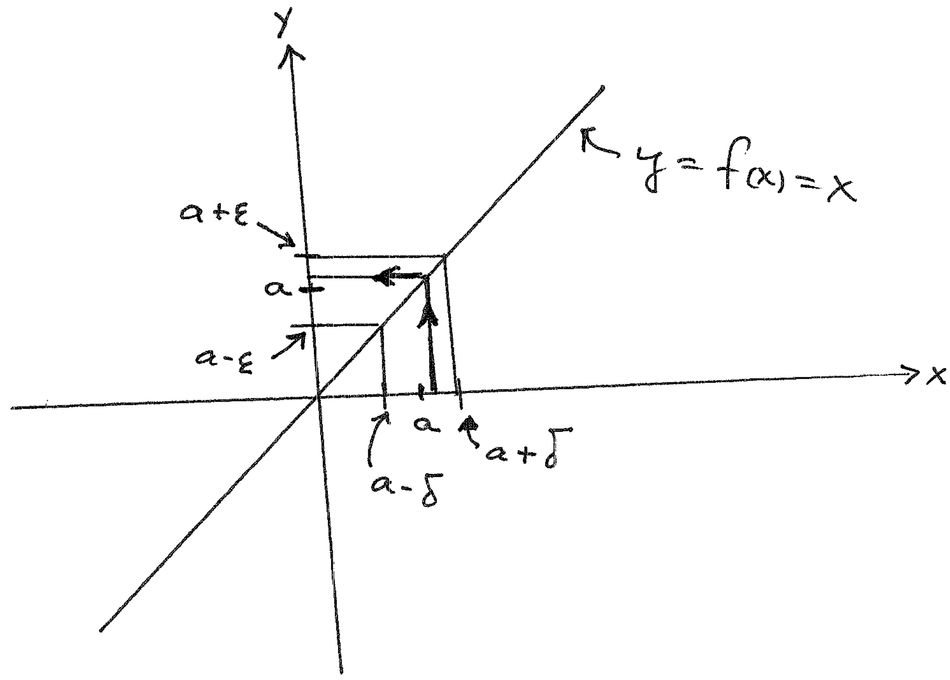
(2) Let's prove that if $f: \mathbb{R} \to \mathbb{R}$ is defined $\forall\, x \in \mathbb{R}$ by

$$f(x) = x$$

then $\forall\, a \in \mathbb{R}$

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a$$

Picture (for the case where $a>0$):



For any given $\varepsilon > 0$ take $\delta = \varepsilon > 0$. Then

$$0 < |x-a| < \delta \implies |f(x) - a| = |x-a| < \delta = \varepsilon .$$

We could continue with such examples — But instead we shall *state* some rules (= facts) about limits which shall enable us to prove a wide range of useful results. The proofs of these results are referred or deferred to a more advanced course though for the sake of eager-beavers I've included sketches of these results in a separate hand out. As far as the present course is concerned you will just be responsible for understanding (& knowing) the careful statements of these results —

First, by way of background, suppose that $f : \mathbb{R} \to \mathbb{R}$ & $g : \mathbb{R} \to \mathbb{R}$ are functions, and that $c \in \mathbb{R}$.
We then define new functions

$$f+g : \mathbb{R} \to \mathbb{R},$$

$$f \cdot g : \mathbb{R} \to \mathbb{R},$$

$$c \cdot g : \mathbb{R} \to \mathbb{R},$$

$$\frac{1}{g} : \mathbb{R} \to \mathbb{R},$$

$$\frac{f}{g} : \mathbb{R} \to \mathbb{R},$$

& $$f \circ g : \mathbb{R} \to \mathbb{R}$$

as follows: $\forall x \in \mathbb{R}$

$$(f+g)(x) := f(x) + g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$(c \cdot g)(x) = c \cdot [g(x)]$$

This is a special case of the above if we think of $c \in \mathbb{R}$ as the constant function $c(x) = c \quad \forall x \in \mathbb{R}$

$$\frac{1}{g}(x) = \frac{1}{g(x)}, \text{ if } g(x) \neq 0.$$

otherwise $\frac{1}{g}$ is not defined!

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \left(f \cdot \frac{1}{g}\right)(x), \text{ if } g(x) \neq 0. \text{ Otherwise } \frac{f}{g} \text{ is not defined!}$$

$$f \circ g(x) = f[g(x)]$$

<u>Theorem:</u> If $\lim_{x \to a} f(x)$ exists & if $\lim_{x \to a} g(x)$ exists then

① $\lim_{x \to a} (f+g)(x)$ exists and

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} [f(x)] + \lim_{x \to a} [g(x)]$$

[i.e., the Limit of a sum is the sum of the limits, provided each exists.]

② $\lim_{x \to a} (f \cdot g)(x)$ exists and

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} [f(x)] \cdot \lim_{x \to a} [g(x)]$$

[i.e., the limit of a product is the product of the limits provided each exists]

③ $\forall c \in \mathbb{R}$

$$\lim_{x \to a} [c \cdot g(x)] = c \cdot \left(\lim_{x \to a} [g(x)] \right)$$

[i.e., the limit of a constant times a function is the constant times the limit of that function if it exists]

④ If $\lim_{x \to a} g(x) \neq 0$, then

$\lim_{x \to a} \left(\frac{1}{g}\right)(x)$ exists & $\lim_{x \to a} \left(\frac{f}{g}\right)(x)$ exists

and

$$\lim_{x \to a} \left[\frac{1}{g(x)}\right] = \frac{1}{\lim_{x \to a} [g(x)]}$$

and

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} [f(x)]}{\lim_{x \to a} [g(x)]}.$$

i.e., the limit of a quotient is the quotient of the limits provided each exists & the limit of the function in the denominator is not zero!

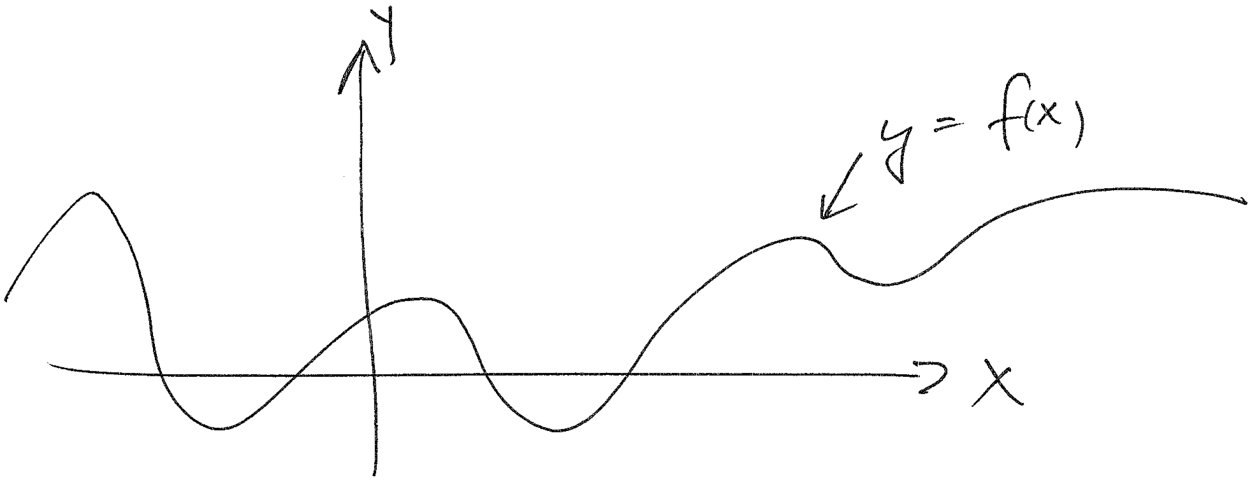
Finally, we can say something about

$$\lim_{x \to a} f \circ g(x)$$

but 1st we need to introduce the notion of a <u>continuous</u> function.

Intuitively, a function $f: \mathbb{R} \to \mathbb{R}$ is continuous if its graph consists of a single unbroken curve - i.e., if you can trace its graph without lifting the pencil from the paper or the chalk from the board.

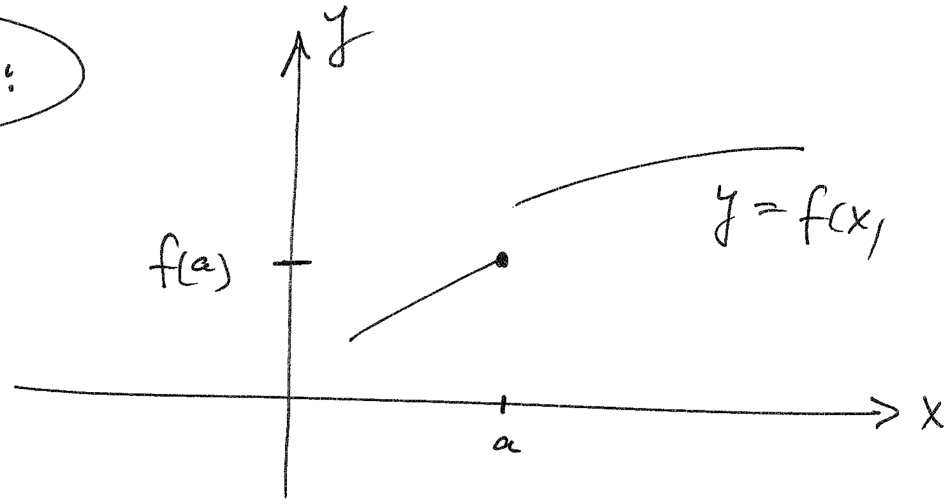
Picture of a continuous function - or rather - the picture of the graph of a continuous function $f: \mathbb{R} \to \mathbb{R}$:



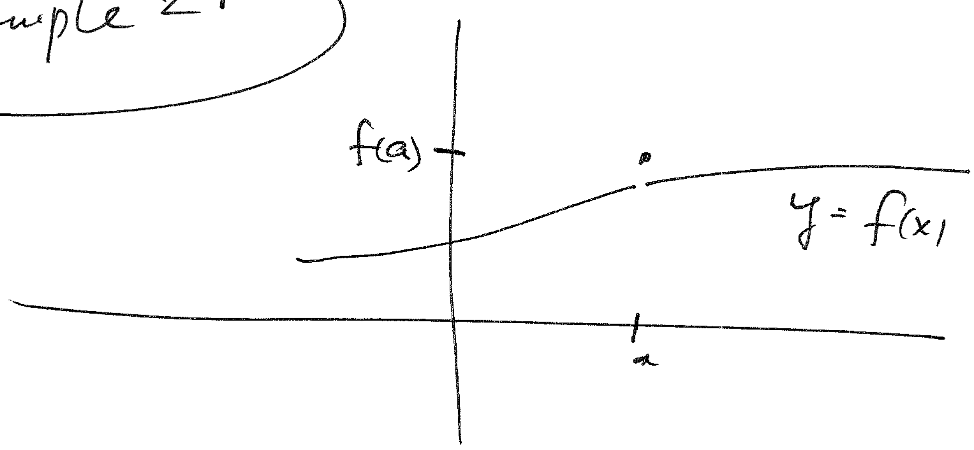
It is therefore clear, intuitively that if $f: \mathbb{R} \to \mathbb{R}$ is continuous, then points close together get sent to points that are close together.

By contrast, here are the graphs of functions $f : \mathbb{R} \to \mathbb{R}$ that are <u>not</u> continuous:

Example 1:



Example 2:



Example 1 has what is called a <u>jump discontinuity</u> while <u>Example 2</u> has what is called a <u>removable discontinuity</u>.

In each case it is false that points arbitrarily close to a get sent to points arbitrarily close to f(a).

In general, if X and Y are subsets of $\mathbb{R}$ & if $f: X \to Y$ is a function, then one defines $f: X \to Y$ to be continuous at $a \in \mathbb{R}$ provided 3 conditions hold:

① f is defined at a

i.e., $f(a)$ exists

i.e., $a \in X$

② $\lim_{x \to a} f(x)$ exists &

③ $\lim_{x \to a} f(x) = f(a)$.

One can actually take condition 3 as the sole defining condition with the understanding that for that condition to make sense

① $f(a)$ must exist

② $\lim_{x \to a} f(x)$ " "

& ③ $\lim_{x \to a} f(x) = f(a)$.

Next, if X & Y are subsets of $\mathbb{R}$ & if $f : X \to Y$ is a function then one defines $f : X \to Y$ to be continuous if it is continuous at each $a \in X$.

Thus, for example, the function

$$f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R} - \{0\}$$

given by

$$f(x) = \frac{1}{x}$$

is continuous wherever it is defined. While it is correct to say that f is not continuous at $x = 0$ because f is not even defined at $x = 0$, it is not appropriate to say that $x = 0$ is a point of discontinuity of f. Rather, if a is a point at which $f : X \to Y$ is defined (i.e., if $a \in X$) then & only then is a a point of discontinuity of f if either $\underset{x \to a}{\text{Limit}}\, f(x)$ does not exist or $\underset{x \to a}{\text{Limit}}\, f(x) \neq f(a)$.

<u>Theorem:</u> Let X, Y & Z be subsets of $\mathbb{R}$ & let

$$g : X \to Y \quad \& \quad f : Y \to Z$$

be functions so that the composite function $f \circ g : X \to Z$ is defined via

$$f \circ g\,(x) = f[g(x)].$$

Picture:

$$\begin{tikzcd} X \arrow[r, "g"] & Y \arrow[dd, "f"] \\ x \arrow[r, mapsto, "g"] \arrow[dr, "f \circ g"'] & g(x) \arrow[d, "f"] \\ & f[g(x)] \in Z \end{tikzcd}$$

Then

$$\lim_{x \to a} f \circ g\,(x) = \lim_{x \to a} f[g(x)]$$

$$= f\left[\lim_{x \to a} g(x)\right]$$

<u>provided</u> that

(1) $b := \lim_{x \to a} g(x)$ exists

and

(2) f is continuous at b.

Theorem: (Squeeze Play Theorem):

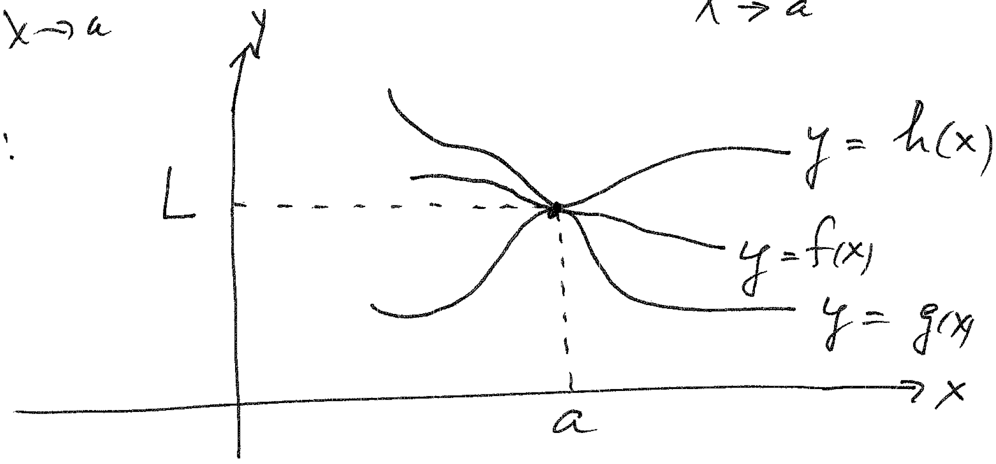
If $\forall x$ near a

$$g(x) \le f(x) \le h(x)$$

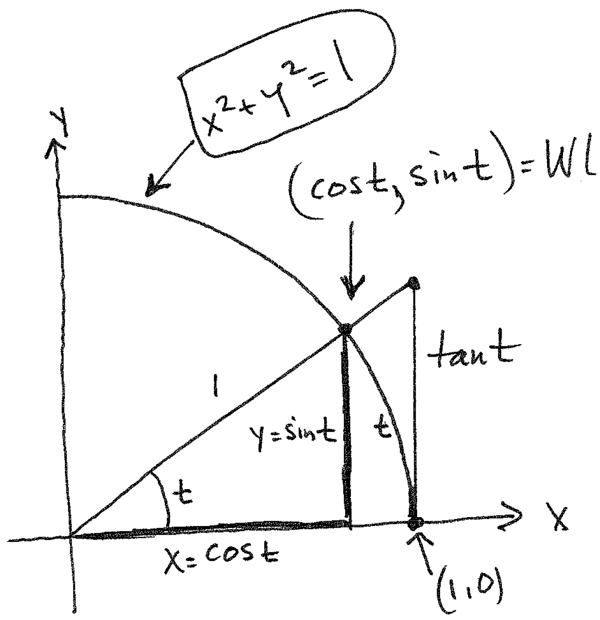
& if $\lim\limits_{x \to a} g(x) = L = \lim\limits_{x \to a} h(x)$

then $\lim\limits_{x \to a} f(x)$ exists and $\lim\limits_{x \to a} f(x) = L$

Picture:



Application: For $0 < t < \frac{\pi}{2}$ consider:



Here $x = \cos t$ while $y = \sin t$.

Clearly $\frac{1}{2} x \cdot y < \frac{1}{2} t < \frac{1}{2} \cdot 1 \cdot \tan t$

i.e.,

$$(*) \quad \frac{1}{2} \cos t \cdot \sin t < \frac{1}{2} t < \frac{1}{2} \frac{\sin t}{\cos t}$$

For $0 < t < \frac{\pi}{2}$, $\frac{1}{2} \sin t > 0$ so we may divide $(*)$ by $\frac{1}{2} \sin t$ to obtain

$$\cos t < \frac{t}{\sin t} < \frac{1}{\cos t}$$

or upon taking reciprocals

$$\cos t < \frac{\sin t}{t} < \frac{1}{\cos t}$$

($2 < 3 < 4 \implies \frac{1}{4} < \frac{1}{3} < \frac{1}{2}$

Since the more pieces you divide a pie into the smaller the pieces.)

Now as $t \to 0$ through values of $t > 0$

$\cos t \to 1$ & $\frac{1}{\cos t} \to 1$

Hence

$$\begin{array}{ccccc} \cos t & < & \frac{\sin t}{t} & < & \frac{1}{\cos t} \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \leq & & \leq & 1 \end{array}$$

So $\displaystyle \lim_{\substack{t \to 0 \\ t > 0}} \frac{\sin t}{t} = 1$

Replacing $t \in (0, \frac{\pi}{2})$ by $-t \in (-\frac{\pi}{2}, 0)$ we find that

$$\cos(-t) < \frac{\sin(-t)}{-t} < \frac{1}{\cos(-t)}$$

But $\sin(-t) = -\sin t$

while $\cos(-t) = \cos t$

so
$$\cos(-t) < \frac{\sin(-t)}{-t} < \frac{1}{\cos(-t)}$$

reduces to
$$\cos t < \frac{-\sin t}{-t} = \frac{\sin t}{t} < \frac{1}{\cos t}$$

& again
$$\lim_{t \to 0^-} \frac{\sin t}{t} = 1$$

i.e.
$$\lim_{\substack{t \to 0 \\ t<0}} \frac{\sin t}{t} = 1 .$$

Remark: In calculus we assume t is a real number or an angle measured in radians as explained in the following

Note: In the above argument

t = the radian measure of the angle

so $\lim_{t \to 0} \frac{\sin t}{t} = 1 = 1$ radian

If the angle were measured in degrees rather than radians the limit would be $\frac{\pi}{180} \approx 0.0174532925199\ldots$

which is what you'd find on your electronic calculator if it is in degree mode rather than radian mode!

Examples of some limits that fail to exist:

① Let $f: \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by

$$f(x) = \frac{|x|}{x}$$ where $|x| :=$ the absolute value of x

defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0. \end{cases}$$

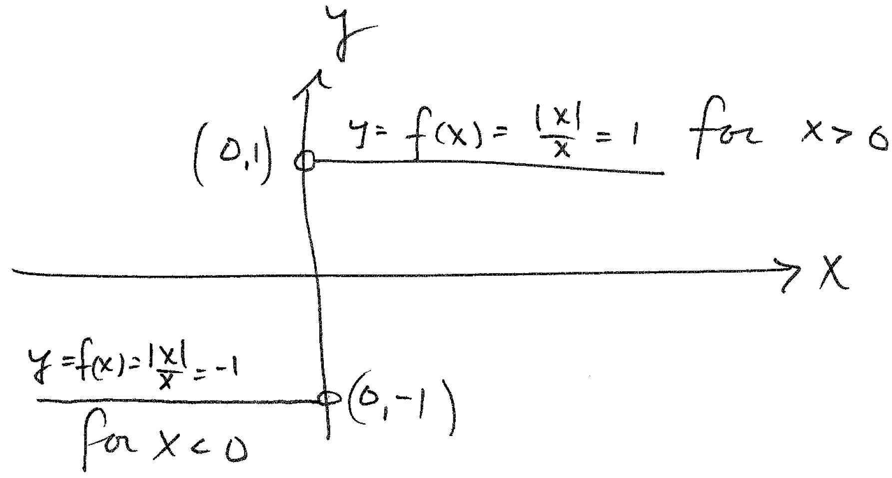
[Thus, for example, $|2| = 2$ since $2 \geq 0$
while $|0| = 0$ since $0 \geq 0$
while $|0| = -0 = 0$ since $0 \leq 0$
while $|-2| = -(-2) = 2$ since $-2 \leq 0$]

Note: Don't be misled into believing that $-x$ is always negative!
Whether $-x$ is positive, negative or zero depends on whether x is negative, positive or zero!

Clearly, $\forall\, x > 0$, $f(x) = \frac{|x|}{x} = \frac{x}{x} = 1$

while $\forall\, x < 0$, $f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1$

Hence, the graph of $f: \mathbb{R} - \{0\} \to \mathbb{R}$ looks like this:



Clearly

$$\operatorname{Limit}_{x \to 0^+} f(x) := \operatorname{Limit}_{\substack{x \to 0 \\ x > 0}} f(x) = \operatorname{Limit}_{\substack{x \to 0 \\ x > 0}} 1 = 1$$

while

$$\operatorname{Limit}_{x \to 0^-} f(x) := \operatorname{Limit}_{\substack{x \to 0 \\ x < 0}} f(x) = \operatorname{Limit}_{\substack{x \to 0 \\ x < 0}} (-1) = -1$$

So $$\operatorname{Limit}_{x \to 0^+} f(x) = 1 \neq -1 = \operatorname{Limit}_{x \to 0^-} f(x)$$

Since the right-hand limit

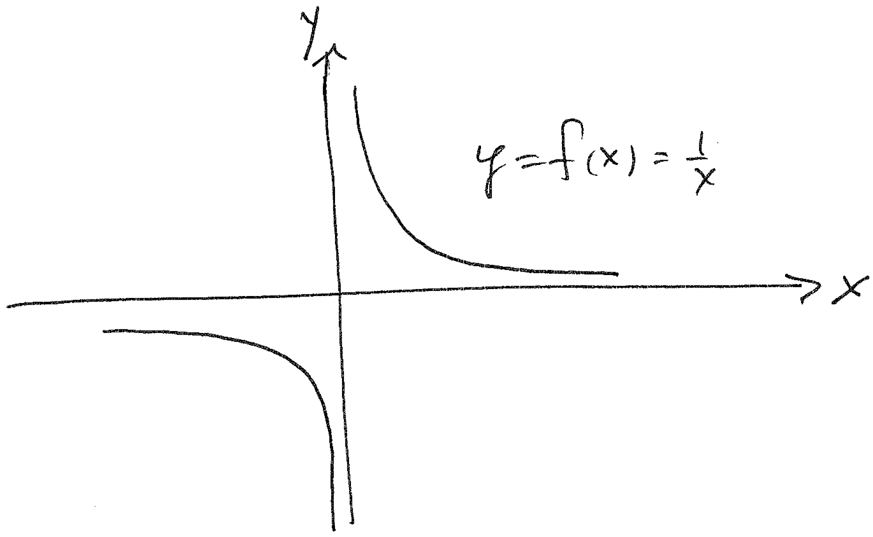
$$\operatorname{Limit}_{x \to 0^+} f(x)$$

differs from the left-hand limit

$$\operatorname{Limit}_{x \to 0^-} f(x)$$

the (2-sided) limit $\operatorname{Limit}_{x \to 0} f(x)$ does **not** exist!

2



Consider the function $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$, pictured above, defined $\forall x \neq 0$ via

$$f(x) := \frac{1}{x}$$

Clearly,

$$\lim_{x \to 0^+} f(x) = +\infty$$

i.e., $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \frac{1}{\lim_{x \to 0^+} x} = \frac{1}{0^+} = +\infty$$

i.e., $$\frac{1}{\substack{\text{Little} \\ \text{positive} \\ \text{number}}} = \text{Big positive number}$$

while $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = \frac{1}{\lim_{x \to 0^-} x} = \frac{1}{0^-} = -\infty$$

i.e., $$\frac{1}{\substack{\text{Little} \\ \text{negative} \\ \text{number}}} = \text{Big negative number}$$

Actually, $+\infty$ is not to be thought of as a Big positive number. To say that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$ is to say that

As x approaches 0 from the positive side of 0

as $x \to 0^+$, $\frac{1}{x}$ gets bigger than any positive integer no matter how large,

for if $x < \frac{1}{N}$, then $\frac{1}{x} > \frac{1}{\frac{1}{N}} = N$

Likewise, $-\infty$ is not to be thought of as a Big negative number.

Rather, to say that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

As x approaches 0 from the negative side of 0

is to say that as $x \to 0^-$, $\frac{1}{x}$ gets smaller than any negative integer no matter how large in absolute value!

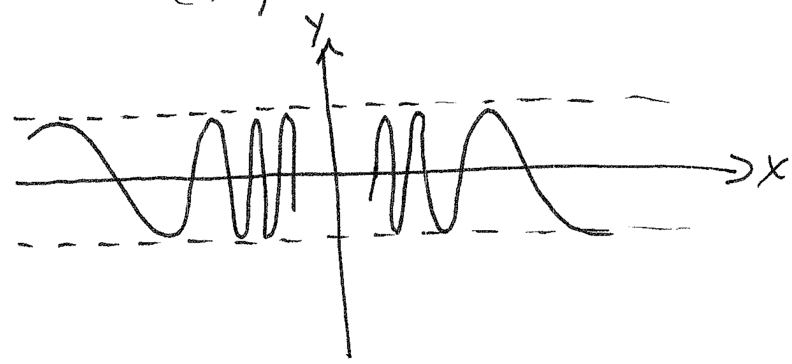
i.e. if $\frac{1}{-N} < x < 0$ for $N > 0$

then $\frac{1}{x} < -N$.

(3) A limit may fail to exist because it oscillates.
For example, Let $f: \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by

$$f(x) = \sin\left(\tfrac{1}{x}\right) \quad \text{for } x \neq 0.$$

Picture:



Here $\displaystyle \text{Limit}_{x \to 0} f(x) = \text{Limit}_{x \to 0} \sin\left(\tfrac{1}{x}\right)$

does not exist since as $x \to 0^+$ $\frac{1}{x} \to +\infty$ & $\sin w$ oscillates as $w \to +\infty$. Likewise, as $x \to 0^-$, $\frac{1}{x} \to -\infty$ & $\sin w$ oscillates (between -1 & 1) as $w \to -\infty$.

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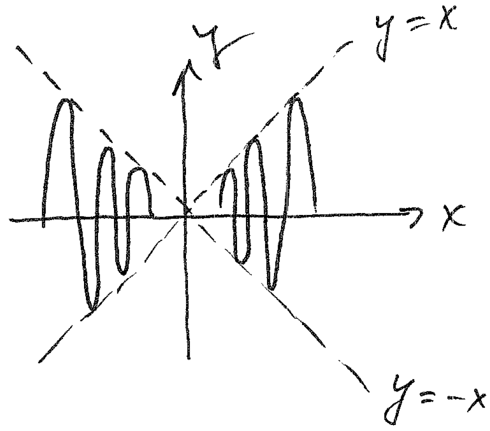
(4) Note, one can remove the difficulty presented in example 3 by "damping" the sine wave. This is done by defining $g: \mathbb{R} - \{0\} \to \mathbb{R}$ via

$$g(x) := x \cdot \sin\left(\tfrac{1}{x}\right) \quad \text{for } x \neq 0$$

Now $|g(x)| = \left| x \cdot \sin\left(\frac{1}{x}\right) \right| = |x| \cdot \left|\sin\left(\frac{1}{x}\right)\right| \le |x| \cdot 1 = |x|$

So the graph of g looks like this:



So even though the graph still oscillates infinitely often the graph lies between the lines $y = x$ & $y = -x$ & so

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0.$$