MATH 2107-601 **EXAM #1** NAME: KEY
(Print)

MONDAY

FEBRUARY 19, 2001 NAME: ____________
(Signature)

[6:00 p.m. - 7:15 p.m.] STUDENT I. D.: ____________

INSTRUCTIONS: There are 2 parts to this exam, the main part plus an optional extra credit part (on the last page). The main part is worth 100 points while the extra credit part is worth 10 points - but any score over 100 will be truncated to 100.

Clarity of exposition is an integral part of a correct solution to any problem.

Good Luck.

PLEASE SIGN THE FOLLOWING STATEMENT:
On my honor, I declare that the work that follows is entirely my own. With regard to all the questions on this exam, I have neither given nor received help from anyone [including myself, say, via any type of cheat sheet or device (e.g., cell phone)]. Nor have I used a calculator of any sort (i.e., regular or programmable).

NAME: ____________
(Signature)

PLEASE DO NOT WRITE BELOW THIS LINE:

20
16
1. a. 3
b. 4
c. 6
d. 3

4
2. a. 1
b. 3

15
11
3. a. 2
b. 1
c. 4
d. 4

4
4. 4

Total: 35

10
5. 10

55
6. a. 5
b. 5
c. 5
d. 5
e. 5
f. 5
g. 5
h. 5
i. 5
j. 5
k. 5

Total: 65

EXTRA-CREDIT:

10
1. 5
2. 5

TOTALS:

Column 3: 10

Column 2: 65

Column 1: 35

Grand Total: 110 → 100

Grade: A+

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1. (a) By definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

[3 points]

(b) Geometrically, $f'(x)$ is equal to the slope of the straight line tangent to the graph of f at the point $(x, f(x))$.

[4 points]

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and that $a \in \mathbb{R}$. Then, by definition,

$$\lim_{x \to a} f(x) = L$$

if and only if for any given (error bound) $\varepsilon > 0$

[6 points] there exists a (deviation bound) $\delta > 0$ so that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

(d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and that $a \in \mathbb{R}$. Then, by definition, f is <u>continuous at</u> a if and only if

[3 points]

(1) f is defined at a,

(2) $\lim_{x \to a} f(x)$ exists,

and (3) $\lim_{x \to a} f(x) = f(a)$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function. What characteristic of the graph of f enables you to tell at a glance

(a) that f is everywhere continuous?

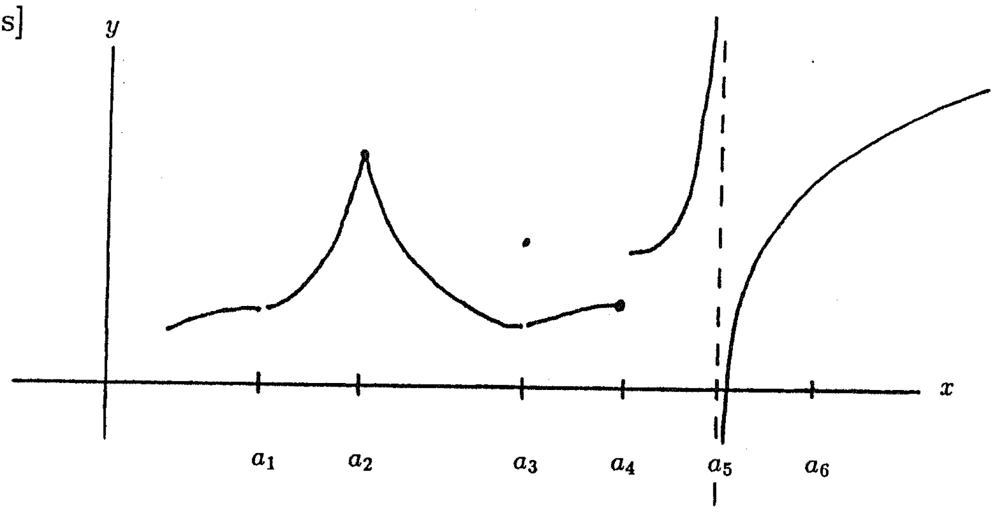
[1 point] The graph of f is unbroken, i.e., it has no jumps or breaks (in it).

(b) that f is everywhere differentiable?

[3 points] The graph of f is unbroken, and has no corners or vertical tangents or wild oscillations.

3. With reference to the graph below,

[11 points]



list all points p from the set $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ for which it is true that

(a) f is <u>continuous</u> at p.	**<u>ANSWER:</u>** a_2, a_6
(b) f is <u>differentiable</u> at p.	**<u>ANSWER:</u>** a_6
(c) $\lim_{x \to p} f(x)$ <u>exists</u>.	**<u>ANSWER:</u>** a_1, a_2, a_3, a_6
(d) f is <u>defined</u> at p.	**<u>ANSWER:</u>** a_2, a_3, a_4, a_6

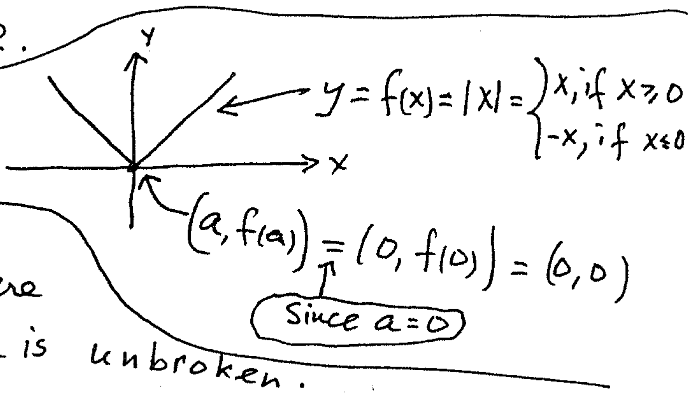
<u>NOTE:</u> Some points may possibly belong to more than one category. BUT, *Nota Bene*, the number of wrong answers will be subtracted from the number of right answers. This is to discourage "padded answers."

4. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a point $a \in \mathbb{R}$ such that f is continuous at a but $f'(a)$ does <u>not</u> exist.

[To get credit, you must give at least some indication of why f is continuous at a and why $f'(a)$ does not exist.]

[4 points] Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x$ for all $x \in \mathbb{R}$, and let $a = 0 \in \mathbb{R}$.

<u>Picture:</u>



Then, $f : \mathbb{R} \to \mathbb{R}$ is everywhere continuous since its graph is unbroken.

Yet $f'(0)$ does not exist because the graph of f has a corner at $(0, f(0)) = (0, 0)$.

5. Compute $f'(x)$ directly from the definition in case

$$f(x) = \sqrt{8x+4}\ .$$

NOTE: No credit will be given for just the answer.

[10 points]

$$f'(x) = \underset{h\to 0}{L} \frac{f(x+h) - f(x)}{h} = \underset{h\to 0}{L} \frac{\sqrt{8(x+h)+4} - \sqrt{8x+4}}{h}$$

$$= \underset{h\to 0}{L} \frac{\sqrt{8(x+h)+4} - \sqrt{8x+4}}{h} \cdot \frac{\sqrt{8(x+h)+4} + \sqrt{8x+4}}{\sqrt{8(x+h)+4} + \sqrt{8x+4}}$$

$$= \underset{h\to 0}{L} \frac{8(x+h)+4 - (8x+4)}{h\cdot\left[\sqrt{8(x+h)+4} + \sqrt{8x+4}\right]}$$

$$= \underset{h\to 0}{L} \frac{8x + 8h + 4 - 8x - 4}{h\cdot\left[\sqrt{8(x+h)+4} + \sqrt{8x+4}\right]}$$

$$= \underset{h\to 0}{L} \frac{8h}{h\cdot\left[\sqrt{8(x+h)+4} + \sqrt{8x+4}\right.}$$

$$= \underset{h\to 0}{L} \frac{8}{\sqrt{8(x+h)+4} + \sqrt{8x+4}}$$

$$= \frac{8}{\sqrt{8x+4} + \sqrt{8x+4}} = \frac{8}{2\cdot\sqrt{8x+4}}$$

$$= \frac{4}{\sqrt{8x+4}} = \frac{4}{\sqrt{4(2x+1)}} = \frac{4}{\sqrt{4}\cdot\sqrt{2x+1}}$$

$$= \frac{4}{2\cdot\sqrt{2x+1}} = \frac{2}{\sqrt{2x+1}}$$

6. Evaluate the following limits. [No work, no credit.]

[55 points: 5 each]

(a) $\lim_{x\to a}\frac{x^4-a^4}{x-a} = \underset{x\to a}{L}\frac{(x^2+a^2)\cdot(x^2-a^2)}{x-a} = \underset{x\to a}{L}\frac{(x^2+a^2)\cdot(x+a)\cdot(x-a)}{x-a}$

$= \underset{x\to a}{L}(x^2+a^2)\cdot(x+a) = \underset{x\to a}{L}(x^3+ax^2+a^2x+a^3)$

$=(a^2+a^2)\cdot(a+a)$

$= 2a^2\cdot 2a = 4a^3$

OR

$a^3+a\cdot a^2+a^2\cdot a+a^3$

$a^3+a^3+a^3+a^3 = 4a^3$

(b) $\lim_{x\to -1}\frac{x^4-1}{x+1} = \underset{x\to -1}{L}(x^2+1)\cdot(x^2-1) = \underset{x\to -1}{L}\frac{(x^2+1)\cdot(x-1)\cdot(x+1)}{x+1}$

$= \underset{x\to -1}{L}(x^2+1)\cdot(x-1) = \underset{x\to -1}{L}(x^3-x^2+x-1)$

$= [(-1)^2+1]\cdot[-1-1]$

$= [1+1]\cdot[-2] = 2\cdot(-2) = -4$

OR

$(-1)^3-(-1)^2+-1-1$

$-1-1-1-1 = -4$

OR

$\left.\underset{x\to a}{L}\frac{x^4-a}{x-a}\right|_{a=-1} = \left.+4a^3\right|_{a=-1}$

$=4(-1)^3=4\cdot(-1)=-4$

(c) $\lim_{x\to 0}\frac{1+\sin x}{\cos x} = \frac{1+0}{1} = \frac{1}{1} = 1$

OR

$= \underset{x\to 0}{L}\frac{1+\sin x}{\cos x}\cdot\frac{1-\sin x}{1-\sin x} = \underset{x\to 0}{L}\frac{1-\sin^2 x}{\cos x\cdot[1-\sin x]}$

$= \underset{x\to 0}{L}\frac{\cos^2 x}{\cos x\cdot[1-\sin x]} = \underset{x\to 0}{L}\frac{\cos x}{\cos x}\cdot\frac{\cos x}{1-\sin x} = 1\cdot\frac{1}{1-0} = 1\cdot 1 = 1$

(d) If $f(x) = |x-3|$, then

$\lim_{h\to 0+}\frac{f(3+h)-f(3)}{h} = \underset{\substack{h\to 0\\ h>0}}{L}\frac{|3+h-3|-|3-3|}{h} = \underset{\substack{h\to 0\\ h>0}}{L}\frac{|h|}{h}$

$= \underset{\substack{h\to 0\\ h>0}}{L}\frac{h}{h} = \underset{\substack{h\to 0\\ h>0}}{L} 1 = 1$

(e) $\lim_{x\to 0}\frac{\tan 9x}{3x} = \underset{x\to 0}{L}\frac{3}{3}\cdot\frac{\tan 9x}{3x} = \underset{x\to 0}{L}\frac{3}{1}\cdot\frac{\tan 9x}{9x} = 3\cdot 1 = 3$

OR

$\underset{x\to 0}{L}\frac{1}{3x}\cdot\frac{\sin 9x}{\cos 9x} = \underset{x\to 0}{L}\frac{9x}{3x}\cdot\frac{\sin 9x}{9x}\cdot\frac{1}{\cos 9x}$

$= \underset{x\to 0}{L}\frac{9}{3}\cdot\frac{\sin 9x}{9x}\cdot\frac{1}{\cos 9x} = \frac{9}{3}\cdot 1\cdot\frac{1}{1} = 3$

(f) $\lim_{x\to 0}\frac{\sec x-1}{x^2} = \underset{x\to 0}{L}\frac{\sec x-1}{x^2}\cdot\frac{\sec x+1}{\sec x+1} = \underset{x\to 0}{L}\frac{\sec^2 x-1}{x^2\cdot[\sec x+1]}$

$= \underset{x\to 0}{L}\frac{\tan^2 x}{x^2}\cdot\frac{1}{\sec x+1} = \underset{x\to 0}{L}\left(\frac{\tan x}{x}\right)^2\cdot\left[\frac{1}{\sec x+1}\right]$

$= 1^2\cdot\left(\frac{1}{1+1}\right) = 1\cdot\frac{1}{2} = \frac{1}{2}$

Note: Writing s for sin x & c for cos x we have that

$s^2+c^2=1$

so $\tan^2 x+1 = \frac{s^2}{c^2}+\frac{c^2}{c^2} = \frac{s^2+c^2}{c^2} = \frac{1}{c^2} = \sec^2 x$.

But then $\tan^2 x+1 = \sec^2 x$ implies that $\sec^2 x-1 = \tan^2 x$.

6. (Continued): Evaluate the following limits. **[No work, no credit.]**

(g) $\lim_{x\to5}\frac{x-5}{x^2-6x+5} = \underset{x\to5}{L}\ \frac{x-5}{(x-5)\cdot(x-1)} = \underset{x\to5}{L}\ \frac{1}{x-1} = \frac{1}{5-1} = \frac{1}{4}$

Note:

$$\begin{array}{r|rrr} 5 & 1 & -6 & 5 \\ & & 5 & -5 \\ \hline & 1 & -1 & \boxed{0} \end{array}$$

$\rightarrow 1\cdot x^1 + -1\cdot x^0 = x-1$

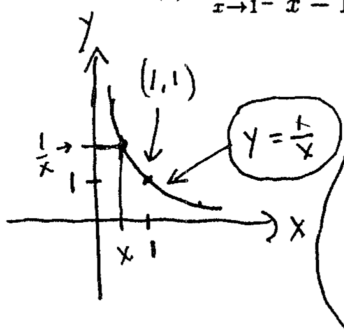
(h) $\lim_{x\to-\infty}\frac{x}{\sqrt{x^2-4x}} = \underset{x\to-\infty}{L}\ \frac{x}{\sqrt{x^2\cdot\left(1-\frac{4}{x}\right)}} = \underset{x\to-\infty}{L}\ \frac{x}{\sqrt{x^2}\cdot\sqrt{1-\frac{4}{x}}}$

$= \underset{x\to-\infty}{L}\ \frac{x}{|x|\cdot\sqrt{1-\frac{4}{x}}} = \underset{x\to-\infty}{L}\ \frac{x}{-x\cdot\sqrt{1-\frac{4}{x}}}$

$= \underset{x\to-\infty}{L}\ \frac{-1}{\sqrt{1-\frac{4}{x}}} = \frac{-1}{\sqrt{1-0}} = \frac{-1}{\sqrt{1}} = \frac{-1}{1} = -1$

(i) $\lim_{x\to+\infty}\frac{4x}{x-1} = \underset{x\to+\infty}{L}\ \frac{4x}{x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \underset{x\to+\infty}{L}\ \frac{4}{1-\frac{1}{x}} = \frac{4}{1-0} = \frac{4}{1} = 4$

(j) $\lim_{x\to1^-}\frac{4x}{x-1} = \underset{\substack{x\to1\\x<1}}{L}\ \frac{4x}{x-1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \underset{\substack{x\to1\\x<1}}{L}\ \frac{4}{1-\frac{1}{x}} = -\infty$



As we see from the graph at the left, for $0<x<1$, $\frac{1}{x}>1$, so $1-\frac{1}{x}<0$, so

smaller ↗ ↖ bigger

since as $x\to1$ with $x<1$ $1-\frac{1}{x}\to0$ with $1-\frac{1}{x}<0$

(k) $\lim_{x\to4}\frac{4-\sqrt{x+12}}{4-x} = \underset{x\to4}{L}\ \frac{4-\sqrt{x+12}}{4-x}\cdot\frac{4+\sqrt{x+12}}{4+\sqrt{x+12}}$

$= \underset{x\to4}{L}\ \frac{16-(x+12)}{(4-x)\cdot\left[4+\sqrt{x+12}\right]} = \underset{x\to4}{L}\ \frac{16-x-12}{(4-x)\cdot\left[4+\sqrt{x+12}\right]}$

$= \underset{x\to4}{L}\ \frac{4-x}{(4-x)\cdot\left[4+\sqrt{x+12}\right]} = \underset{x\to4}{L}\ \frac{1}{4+\sqrt{x+12}}$

$= \frac{1}{4+\sqrt{4+12}} = \frac{1}{4+\sqrt{16}} = \frac{1}{4+4} = \frac{1}{8}$.

EXTRA-CREDIT: [**10 points**: **But, any score over 100 will be truncated to 100.**]

1. Given $\varepsilon = .01$, find $\delta > 0$ so that $|(4x+5)-17| < .01$ whenever $0 < |x-3| < \delta$.

[5 points]

SOLUTION:

$$|(4x+5)-17|$$

$$|4x+5-17| = |4x-12| = |4\cdot(x-3)| = |4|\cdot|x-3| = 4\cdot|x-3|$$

We want $4\cdot|x-3| = |(4x+5)-17| < .01 = \varepsilon$.

But $4\cdot|x-3| < \varepsilon \iff |x-3| < \frac{\varepsilon}{4}$

so pick $\delta := \frac{\varepsilon}{4} = \frac{.01}{4} = .0025$

$$\begin{array}{r} .0025 \\ 4\overline{).0100} \\ 8 \\ \hline 20 \\ 20 \\ \hline 0 \end{array}$$

Then

$$0 < |x-3| < \delta := \frac{\varepsilon}{4} = \frac{.01}{4} = .0025 \Longrightarrow |(4x+5)-17| = 4\cdot|x-3|$$
$$< 4\cdot\frac{\varepsilon}{4} = \varepsilon$$

2. Find the equation of the straight line that is tangent to the graph of the function

$$f(x) = x^4 + x^3 + x^2 + x + 2 \Longrightarrow f(1) = 1^4 + 1^3 + 1^2 + 1 + 2$$
$$= 1 + 1 + 1 + 1 + 2$$
$$= 4 + 2 = 6$$

at the point (1, 6).

[5 points]

SOLUTION:

$$\Downarrow$$

$$f'(x) = 4\cdot x^3 + 3\cdot x^2 + 2\cdot x^{1} + 1 + 0$$

$$\Downarrow$$

$$f'(1) = 4\cdot 1^3 + 3\cdot 1^2 + 2\cdot 1^{1} + 1$$
$$= 4 + 3 + 2 + 1$$
$$= 7 + 3 = 10$$

$\therefore$ $10 = f'(1) =$ the slope of the straight line tangent to the graph of $y = f(x) = x^4 + x^3 + x^2 + x + 2$ at the point $(1, f(1)) = (1, 6)$,

so $\frac{y-6}{x-1} = 10 \Longrightarrow y - 6 = 10\cdot(x-1) = 10\cdot x - 10$

$$\Downarrow$$

$$y = 10x - 10 + 6 = 10x - 4$$

$\boxed{\text{DR}}$ $y = mx + b = 10x + b$

so $6 = 10\cdot 1 + b$

$\therefore$ $b = 6 - 10 = -4 \Longrightarrow y = 10x + b = 10x - 4.$