MATH 2107-601 **EXAM #2** NAME: KEY
(Print)

WEDNESDAY

NAME: ____________________
(Signature)

October 29, 2003

[6:00 p.m. - 7:15 p.m.] STUDENT I. D.: ____________________

INSTRUCTIONS: There are 2 parts to this exam, the main part plus an optional **EXTRA CREDIT** part (on the last page). The main part is worth 100 points while the extra credit part is worth 12 points. But any score over 100 will be truncated to 100.

Clarity of exposition (including proper spelling and punctuation is an integral part of a correct solution to any problem. In particular, **DO NOT PUT EQUAL SIGNS BETWEEN THINGS THAT ARE NOT EQUAL.** But do put them where they belong.

It is necessary to show all your work.

GOOD LUCK!

PLEASE SIGN THE FOLLOWING STATEMENT:

On my honor, I declare that the work that follows is entirely my own. With regard to all the questions on this exam, I have neither given nor received help from anyone [including myself, say, via any type of cheat sheet or device (e.g., cell phone)]. Nor have I used a calculator of any sort (i.e., regular or programmable).

NAME: ____________________
(Signature)

PLEASE DO NOT WRITE BELOW THIS LINE:

15	1. a. 10	5. a. 5	**EXTRA-CREDIT:**	
	b. 5	b. 5	1.	8
10	2. a. 5	c. 5	2.	4
	b. 5	d. 5	**TOTALS:**	
15	3. 15	e. 5	Column 3:	12
20	4. a. 5	f. 5	Column 2:	40
	b. 5	g. 5	Column 1:	60
	c. 5	h. 5	**Grand Total:**	112 ↘ 100
	d. 5		**Grade:**	A+
	Total: 60	Total: 40		

1. (a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function and let $a\in\mathbb{R}$. Prove that f is continuous at a whenever $f'(a)$ exists.

[10 points]

$$\lim_{h\to 0} f(a+h) = \lim_{h\to 0}\left\{h\cdot\left[\frac{f(a+h)-f(a)}{h}\right] + f(a)\right\}$$

$$= \lim_{h\to 0} h \cdot \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right) + \lim_{h\to 0} f(a)$$

$$= 0\cdot f'(a) + f(a)$$

$$= 0 + f(a) = f(a).$$

(b) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ and a point $a\in\mathbb{R}$ such that f is continuous at a but $f'(a)$ does <u>not</u> exist.

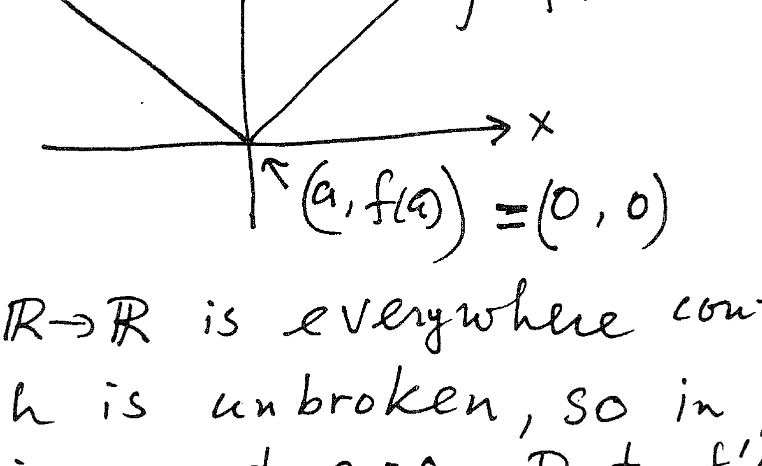
[To get credit, you must give at least some indication of why f is continuous at a and why $f'(a)$ does not exist.]

[5 points] Let $f:\mathbb{R}\to\mathbb{R}$ be defined by setting

$$f(x) := |x| := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

and let $a := 0 \in \mathbb{R}$

Picture:



Then $f:\mathbb{R}\to\mathbb{R}$ is everywhere continuous since its graph is unbroken, so in particular f is continuous at $a=0$. But $f'(0)$ does not exist since were $f'(0)$ to exist, $f'(0)$ would equal the slope of the straight line tangent to the graph of f at $(0,0)$. But the graph of f has a corner at $(0, f(0))$ & so has no tangent there!

2. (a) State the **Chain Rule**. Be sure to state the hypotheses as well as the conclusion.

ANSWER:

The Chain Rule:

Hypotheses:

[1 point] $f'[g(x)]$ exists

and $g'(x)$ exists

Conclusions:

[4 points] $(f \circ g)'(x)$ exists

and $(f \circ g)'(x) = f'[g(x)] \cdot g'(x)$

(b) Complete the following:

[5 points: 1 each]

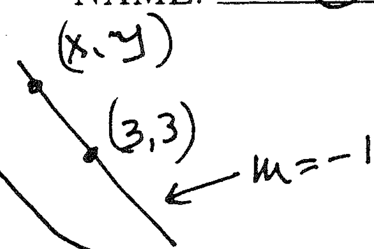
$(F \cdot S)' = F \cdot S' + S \cdot F'$

$\left(\frac{T}{B}\right)' = \frac{B \cdot T' - T \cdot B'}{B^2}$

$(c \cdot F)' = c \cdot F'$

$(F + G)' = F' + G'$

$(F^n)' = n \cdot F^{n-1} \cdot F'$



3. Assume that the equation

$$x^4 y + y^4 x = 486$$

defines y as a differentiable function of x. Use implicit differentiation to find $\frac{dy}{dx}$. Then find the equation of the straight line tangent to the graph of $x^4 y + y^4 x = 486$ at the point $(3,3)$.

[15 points]

SOLUTION:

$$486 = x^4 \cdot y + y^4 \cdot x$$

$$\Downarrow$$

$$0 = \frac{d(486)}{dx} = \frac{d}{dx}\left(x^4 \cdot y + y^4 \cdot x\right)$$

$$= x^4 \cdot \frac{dy}{dx} + y \cdot 4x^3 + y^4 \cdot 1 + x \cdot 4y^3 \frac{dy}{dx}$$

$$\frac{y-3}{x-3} = m = -1$$

$$\Downarrow$$

$$y-3 = -1(x-3) = -x+3$$

$$\Downarrow$$

$$y = -x+6$$

i.e.,

$$x^4 \cdot y + y^4 \cdot x = 486$$

$$\Downarrow$$

$$x^4 \cdot \frac{dy}{dx} + y \cdot 4x^3 + y^4 \cdot 1 + x \cdot 4y^3 \frac{dy}{dx} = 0$$

$$\Downarrow$$

$$\left(x^4 + 4 \cdot x \cdot y^3\right)\frac{dy}{dx} = -\left(y^4 + 4 \cdot y \cdot x^3\right)$$

$$\Downarrow$$

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(3,3)} = \left.\frac{-\left[y^4 + 4 \cdot y \cdot x^3\right]}{x^4 + 4 \cdot x \cdot y^3}\right|_{(x,y)=(3,3)} = \frac{-\left(3^4 + 4 \cdot 3 \cdot 3^3\right)}{3^4 + 4 \cdot 3 \cdot 3^3} = -1$$

$$y = m \cdot x + b = -1 \cdot x + b \Rightarrow b = x + y\Big|_{x=y=3} = 3+3 = 6$$

$$\Downarrow$$

$$y = mx + b = -1 \cdot x + 6$$

or (see the top)

4. For each of the following, use the various rules that we have developed [not the definition] to compute $f'(x)$ when f(x) is given.

Then **SIMPLIFY** your answers as much as possible.

NOTE: DO NOT COMPUTE $f'(x)$ DIRECTLY FROM THE DEFINITION!

[20 points: 5 each]

a. $f(x) = \dfrac{x^3}{x^3+1}$

$$\frac{x^3+1-1}{x^3+1} = \frac{x^3+1}{x^3+1} + \frac{-1}{x^3+1} = 1+(-1)\cdot(x^3+1)^{-1}$$

$$f'(x) = 0+(-1)(-1)\cdot(x^3+1)^{-2}\cdot 3x^2 = \frac{3x^2}{(x^3+1)^2}$$

$\boxed{\text{OR}}$

$$f'(x) = \frac{(x^3+1)\cdot 3x^2 - x^3\cdot 3x^2}{(x^3+1)^2} = \frac{3x^5+3\cdot x^2-3x^5}{(x^3+1)^2} = \frac{3x^2}{(x^3+1)^2}.$$

b. $f(x) = \left(\dfrac{x^4-1}{x^4+1}\right)^5$

$$f'(x) = 5\cdot\left(\frac{x^4-1}{x^4+1}\right)^4\cdot\left[\frac{(x^4+1)\cdot 4x^3-(x^4-1)\cdot 4x^3}{(x^4+1)^2}\right]$$

$$= \frac{5\cdot(x^4-1)^4}{(x^4+1)^4}\cdot\left[\frac{4x^7+4x^3-4x^7+4x^3}{(x^4+1)^2}\right] = \frac{5(x^4-1)^4\cdot 8x^3}{(x^4+1)^6}$$

$$= \frac{40x^3\cdot(x^4-1)^4}{(x^4+1)^6}$$

c. $f(x) = \dfrac{\cos x}{1-\sin x}$

$$f'(x) = \frac{(1-\sin x)\cdot(-\sin x)-\cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x+\sin^2 x+\cos^2 x}{(1-\sin x)^2} = \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x}$$

d. $f(x) = 4x^2\cdot\sin(2x) + 4x\cdot\cos(2x) - 2\cdot\sin(2x)$

$$f'(x) = 4x^2\cdot\cos(2x)\cdot 2 + \sin(2x)\cdot 8x + 4x\cdot[-\sin(2x)\cdot 2] + \cos(2x)\cdot 4 - 2\cdot\cos(2x)\cdot 2$$

$$= 8x^2\cdot\cos(2x) + 0 + 0$$

$$= 8x^2\cdot\cos(2x)$$

5. For each of the following, use the various rules that we have developed [not the definition] to compute $f'(x)$ when f(x) is given.

Do **NOT** simplify your answers.

NOTE: DO NOT COMPUTE $f'(x)$ DIRECTLY FROM THE DEFINITION!

[40 points: 5 each]

a. $f(x) = \cos^8(\csc(x^5)) = \left(\cos\left[\csc\left(x^5\right)\right]\right)^8$

$$f'(x) = 8\cdot\left(\cos\left[\csc\left(x^5\right)\right]\right)^7\cdot\left(-\sin\left[\csc\left(x^5\right)\right]\cdot\left(-\csc\left(x^5\right)\cdot\cot\left(x^5\right)\cdot 5x^4\right)\right)$$

b. $f(x) = \tan\sqrt{x} + \sqrt{\tan x} = \tan\left(x^{1/2}\right) + \left(\tan x\right)^{1/2}$

$$f'(x) = \sec^2\left(x^{1/2}\right)\cdot\frac{1}{2}\cdot x^{-1/2} + \frac{1}{2}\left(\tan x\right)^{-1/2}\cdot\sec^2 x$$

$$= \frac{\sec^2(\sqrt{x})}{2\sqrt{x}} + \frac{\sec^2 x}{2\sqrt{\tan x}}$$

c. $f(x) = \sqrt{1+\sqrt{1+4x}} = \left[1 + (1+4x)^{1/2}\right]^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}\left[1+(1+4x)^{1/2}\right]^{-\frac{1}{2}}\cdot\left[0 + \frac{1}{2}(1+4x)^{-\frac{1}{2}}\cdot 4\right]$$

$$= \frac{1}{2\sqrt{1+\sqrt{1+4x}}}\cdot\frac{4}{2\sqrt{1+4x}} = \frac{1}{\sqrt{1+\sqrt{1+4x}}\cdot\sqrt{1+4x}}$$

$$= \frac{1}{\sqrt{\left(1+\sqrt{1+4x}\right)\cdot(1+4x)}}$$

$$= \frac{1}{\sqrt{1+4x+(1+4x)^{3/2}}}$$

d. $f(x) = x^4\cdot\sin\left(\frac{1}{x^3}\right)$

$= x^4\cdot\sin\left(x^{-3}\right)$

$$f'(x) = x^4\cdot\cos\left(x^{-3}\right)\cdot -3\cdot x^{-4} + \sin\left(x^{-3}\right)\cdot 4x^3$$

$$= 4x^3\cdot\sin\left(\frac{1}{x^3}\right) - 3\cdot\cos\left(\frac{1}{x^3}\right)$$

5. (Continued):

e. $f(x) = \sec x \cdot \cot x^8$

$$f'(x) = \sec(x) \cdot \left[-\csc^2(x^8) \cdot 8x^7\right] + \cot(x^8) \cdot \sec(x) \cdot \tan(x).$$

f. $f(x) = \sec(\cot x^8)$

$$f'(x) = \sec\left[\cot(x^8)\right] \cdot \tan\left[\cot(x^8)\right] \cdot \left(-\csc^2(x^8) \cdot 8x^7\right)$$

$$f'(x) = \frac{1}{5} \cdot 7x^6 + 5 \cdot (-7) \cdot x^{-8} + \frac{1}{7}(x^5)^{-6/7} \cdot 5x^4 + -\frac{1}{5}(x^7)^{-6/5} \cdot 7x^6$$

g. $f(x) = \frac{x^7}{5} + \frac{5}{x^7} + \sqrt[7]{x^5} + \frac{1}{\sqrt[5]{x^7}} + 7 \cdot 5^6$

$$= \frac{1}{5} \cdot x^7 + 5 \cdot x^{-7} + (x^5)^{1/7} + \frac{1}{(x^7)^{1/5}} + 7 \cdot 5^6$$

$$= \frac{1}{5} x^7 + 5 \cdot x^{-7} + x^{5/7} + x^{-7/5} + 7 \cdot 5^6$$

$$f'(x) = \frac{7}{5} x^6 - 35 \cdot x^{-8} + \frac{5}{7} x^{-2/7} - \frac{7}{5} x^{-12/5} + 0$$

Note: $(x^5)^{-6/7} x^4 = x^{\frac{-30}{7}} \cdot x^{\frac{4 \cdot 7}{7}} = x^{-2/7}$

and

$(x^7)^{-6/5} \cdot x^6 = x^{\frac{-42}{5}} \cdot x^{\frac{6 \cdot 5}{5}} = x^{-12/5}$

h. $f(x) = \sqrt[5]{\tan(x^9)}$

$$= \left[\tan(x^9)\right]^{1/5}$$

$$f'(x) = \frac{1}{5}\left[\tan(x^9)\right]^{-4/5} \cdot \left(\sec^2(x^9) \cdot 9x^8\right)$$

$$= \frac{9x^8 \cdot \sec^2(x^9)}{5\left[\tan(x^9)\right]^{4/5}} = \frac{9x^8 \cdot \sec^2(x^9)}{5 \cdot \sqrt[5]{\tan^4(x^9)}}$$

EXTRA-CREDIT: *Circle* the *one* correct choice.

[**12 points** : **But**, **any exam score over 100 will be truncated to 100**.]

1. If $f(x) = \frac{1-x^3}{(x^3+1)^2}$, then $f'(x)$ equals:

   a) $\frac{3x^2(3-x^3)}{(x^3+1)^3}$

   b) $\frac{-1}{2(x^3+1)}$

   c) $\frac{3x^2(x^3-3)}{(x^3+1)^3}$

   d) $\frac{-3x^2}{2(x^3+1)}$

   e) none of the above.

   WORK: [NOTE: You must show your work to get any credit.]

[8 points]

$$f(x) = \frac{1-x^3}{(x^3+1)^2}$$

$$f'(x) = \frac{(x^3+1)^2\cdot(-3x^2) - (1-x^3)\cdot 2(x^3+1)'\cdot 3x^2}{[(x^3+1)^2]^2}$$

$$= \frac{-3x^2\cdot(x^3+1)'\cdot[(x^3+1)' + 2\cdot(1-x^3)]}{(x^3+1)^4}$$

$$= \frac{-3x^2\cdot[x^3+1+2-2x^3]}{(x^3+1)^3} = \frac{-3x^2[3-x^3]}{(x^3+1)^3}$$

$$= \frac{3x^2(x^3-3)}{(x^3+1)^3}.$$

2. If $g(5) = 37$, $g'(5) = \frac{1}{4}$, and $f'(37) = 32$, find $(f\circ g)'(5)$.

   SOLUTION: [NOTE: You must show your work to get any credit.]

[4 points]

ANSWER: $(f\circ g)'(5) = f'[g(5)]\cdot g'(5)$

$$= f'(37)\cdot\frac{1}{4}$$

$$= 32\cdot\frac{1}{4} = 8$$