Math 2107

II. a. $\lim_{x \to a} \frac{x^3 - a^3}{x - a} = ?$

$$x^3 - a^3 = (x-a)\cdot(x^2a^0 + x^1a^1 + x^0a^2)$$
$$= (x-a)\cdot(x^2 + ax + a^2)$$

Alternatively, if $p(x)$ is a polynomial and if a is a real number

then $p(a) = 0 \iff p(x) = (x-a)\cdot q(x)$ for some polynomial $q(x)$

i.e., a is a root (or zero) of $p(x)$

$\iff$ $x - a$ is a factor of $p(x)$.

Set $p(x) = x^3 - a^3$

Then $p(a) = a^3 - a^3 = 0$

so $p(x) = (x-a)\cdot q(x)$.

We may determine $q(x)$ by Synthetic division as follows:

Write $p(x) = x^3 - a^3$ in terms of descending powers of x; i.e., write

$$p(x) = 1\cdot x^3 + 0\cdot x^2 + 0\cdot x^1 + -a^3\cdot x^0$$

Next write down the coefficients of $p(x)$ in a orderly sequence as follows: To

$$p(x) = 1 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x' + -a^3 \cdot x^0$$

associate

$$1 \quad 0 \quad 0 \quad -a^3$$

Next, write down

$$a \,\Big|\, \overline{1 \quad 0 \quad 0 \quad -a^3}$$

from $x - a$

Now procede as follows:

Start

$$\begin{array}{c|cccc} a & 1 & 0 & 0 & -a^3 \\ & & a & a^2 & a^3 \\ \hline & 1 & a & a^2 & \boxed{0} \end{array}$$

$0 = p(a)$

Next to the sequence

$$1 \quad a \quad a^2$$

associate

$$1 \cdot x^2 + a \cdot x' + a^2 \cdot x^0 = x^2 + ax + a^2$$

Then

$$p(x) = x^3 - a^3 = (x - a) \cdot (x^2 + ax + a^2)$$

Check:

$$\begin{array}{r} x^2 + ax + a^2 \\ x - a \\ \hline x^3 + ax^2 + a^2 x \\ -ax^2 - a^2 x - a^3 \\ \hline x^3 + 0x^2 + 0x - a^3 \end{array} = x^3 - a^3 \checkmark$$

Back to the salt mines!

ⓐ $$\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} \frac{(x-a)\cdot(x^2 + a\cdot x + a^2)}{x - a}$$

$$= \lim_{x \to a} (x^2 + a\cdot x + a^2)$$

$$= a^2 + a\cdot a + a^2$$

$$= a^2 + a^2 + a^2 = 3a^2$$

ⓑ $$\lim_{x \to 0} \frac{\tan(7x)}{3x} = \lim_{x \to 0} \frac{\sin(7x)}{\cos(7x)} \cdot \frac{1}{3x}$$

$$= \lim_{x \to 0} \frac{\sin(7x)}{7x} \cdot \frac{7}{3} \cdot \frac{1}{\cos(7x)}$$

$$= \lim_{x \to 0} 1 \cdot \frac{7}{3} \cdot \frac{1}{1}$$

$$= \frac{7}{3}$$

$\cos(7\cdot 0) = \cos(0) = 1$

Note: BEWARE

$$\tan(7x) \neq 7\cdot\tan x$$

So if you write

$$\frac{\tan(7x)}{3x} = \frac{7\tan x}{3x} = \frac{7}{3x}\,\frac{\sin x}{\cos x}$$

$$= \frac{7}{3}\cdot\frac{\sin x}{x}\cdot\frac{1}{\cos x}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\frac{7}{3}\cdot 1\cdot\frac{1}{1} = \frac{7}{3}$$

you will get absolutely no credit even though the answer by this wrong method happens, by coincidence, to be correct!

What is ok is to remark that

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{\cos x}\cdot\frac{1}{x}$$

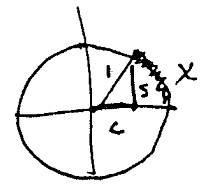
$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}$$

$$= 1\cdot\frac{1}{1} = 1$$

& so $\displaystyle\lim_{x\to 0}\frac{\tan 7x}{3x} = \lim_{x\to 0}\frac{7}{3}\,\frac{\tan 7x}{7x} = \frac{7}{3}\cdot 1 = \frac{7}{3}$

c. $\lim_{x \to 0} \frac{\sec x - 1}{x} = ?$

$\sin^2 x + \cos^2 x = 1$



$\Downarrow$

$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

$\parallel$

$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}$

$\parallel$

$\tan^2 x + 1$

$\therefore \quad \sin^2 x + \cos^2 x = 1 \implies \tan^2 x + 1 = \sec^2 x$

$\implies \sec^2 x - 1 = \tan^2 x$

$\therefore \quad \lim_{x \to 0} \frac{\sec x - 1}{x} = \lim_{x \to 0} \frac{\sec x - 1}{x} \cdot \frac{\sec x + 1}{\sec x + 1}$

$= \lim_{x \to 0} \frac{\sec^2 x - 1}{x \cdot (\sec x + 1)}$

$= \lim_{x \to 0} \frac{\tan^2 x}{x} \cdot \frac{1}{\sec x + 1}$

$= \lim_{x \to 0} \frac{\tan x}{x} \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\left(\frac{1}{\cos x} + 1\right)}$

$= \lim_{x \to 0} \frac{\tan x}{x} \cdot \frac{\sin x}{1 + \cos x}$

$= 1 \cdot \frac{0}{1+1} = 1 \cdot \frac{0}{2} = 1 \cdot 0 = 0$

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d) $\lim\limits_{x \to -\infty} \dfrac{x+1}{\sqrt{x^2+1}} = \lim\limits_{x \to -\infty} \dfrac{x+1}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}$

$$= \lim_{x \to -\infty} \frac{x+1}{\sqrt{x^2} \cdot \sqrt{1+\frac{1}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{x+1}{|x| \cdot \sqrt{1+\frac{1}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{x+1}{-x \cdot \sqrt{1+\frac{1}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{\frac{1}{x} \cdot (x+1)}{\frac{1}{x}(-x)\sqrt{1+\frac{1}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{1+\frac{1}{x}}{-\sqrt{1+\frac{1}{x^2}}}$$

$$= \frac{1+0}{-\sqrt{1+0}} = \frac{1}{-\sqrt{1}}$$

$$= \frac{1}{-1} = -1.$$

e $\lim_{x \to 2} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x \to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)}$

$= \lim_{x \to 2} \frac{x+3}{x-1}$

$= \frac{2+3}{2-1} = \frac{5}{1} = 5$

Note: If $p(x) = x^2 + x - 6$

$p(2) = 2^2 + 2 - 6 = 4 + 2 - 6 = 0$

so $p(x) = (x-2) \cdot q(x)$

To find $q(x)$, write:

$1 \cdot x^2 + 1 \cdot x + -6$

$\downarrow \quad \downarrow \quad \downarrow$

$1 \quad 1 \quad -6$

Then

$$\begin{array}{r|rrr} 2 & 1 & 1 & -6 \\ & & 2 & 6 \\ \hline & 1 & 3 & \boxed{0} \end{array} = p(2)$$

$1 \quad 3$

$\downarrow \quad \downarrow$

$1 \cdot x^1 + 3 \cdot x^0 = x + 3$

so $x^2 + x - 6 = (x-2)(x+3)$

Similarly, if we let

$$p(x) = x^2 - 3x + 2$$

then $p(2) = 2^2 - 3\cdot 2 + 2 = 4 - 6 + 2 = 0$

So $p(x) = (x-2)\cdot q(x)$.

To find $q(x)$, write

$$p(x) = 1\cdot x^2 + -3\cdot x + 2$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 1 & -3 & 2 \end{array}$$

Then

$$\begin{array}{r|rrr} 2 & 1 & -3 & 2 \\ & & 2 & 2 \\ \hline & 1 & -1 & \boxed{0} = p(2) \end{array}$$

$$\begin{array}{cc} \downarrow & \downarrow \\ 1\cdot x^1 + & -1\cdot x^0 \end{array}$$

So $x^2 - 3x + 2 = (x-2)(x-1)$

(f) $$\lim_{x\to 0} \frac{x}{\sin(2x)} = \lim_{x\to 0} \frac{x}{\sin(2x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{1}{\frac{\sin(2x)}{x}}$$

$$= \lim_{x\to 0} \frac{1}{2 \cdot \frac{\sin(2x)}{2\cdot x}}$$

$$= \frac{1}{\lim_{x\to 0} 2 \frac{\sin(2x)}{2x}}$$

$$= \frac{1}{2\cdot 1} = \frac{1}{2}$$

(g) $$\lim_{x\to\infty} \frac{3x-5}{\sqrt{4x^2+7x-5}} = \lim_{x\to\infty} \frac{3x-5}{\sqrt{4x^2\left(1+\frac{7x}{4x^2}-\frac{5}{4x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{3x-5}{\sqrt{(2x)^2}\cdot\sqrt{1+\frac{7}{4x}-\frac{5}{4x^2}}}$$

$$= \lim_{x\to\infty} \frac{3x-5}{|2x|\cdot\sqrt{1+\frac{7}{4x}-\frac{5}{4x^2}}}$$

$$= \lim_{x\to\infty} \frac{3x-5}{|2|\cdot|x|} \frac{1}{\sqrt{1+\frac{7}{4x}-\frac{5}{4x^2}}}$$

$$= \lim_{x \to \infty} \frac{3x - 5}{2 \cdot |x| \cdot \sqrt{1 + \frac{7}{4x} - \frac{5}{4x^2}}}$$

$$= \lim_{x \to \infty} \frac{3x - 5}{2x \cdot \sqrt{1 + \frac{7}{4x} - \frac{5}{4x^2}}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x} \cdot (3x - 5)}{\frac{1}{x} \cdot 2x \cdot \sqrt{1 + \frac{7}{4x} - \frac{5}{4x^2}}}$$

$$= \lim_{x \to \infty} \frac{3 - \frac{5}{x}}{2 \cdot \sqrt{1 + \frac{7}{4x} - \frac{5}{4x^2}}}$$

$$= \frac{3 - 0}{2 \cdot \sqrt{1 + 0 - 0}} = \frac{3}{2 \cdot \sqrt{1}}$$

$$= \frac{3}{2 \cdot 1} = \frac{3}{2}$$

h) $$\lim_{x \to 2^+} \frac{1}{2-x} = \lim_{\substack{x \to 2 \\ x > 2}} \frac{1}{2-x} = ?$$

For $x > 2$, $2-x < 0$

smaller

bigger

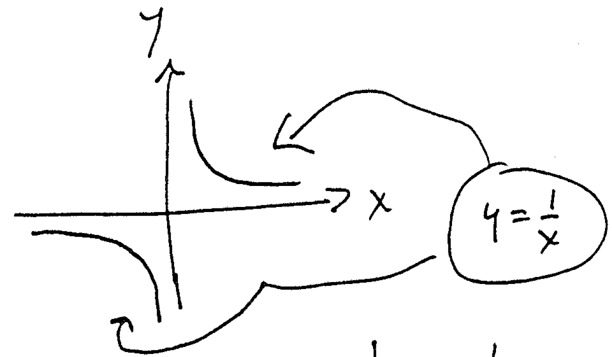
Hence for all $x > 2$, $2-x < 0$

so $\frac{1}{2-x} < 0$ since the reciprocal of a negative number is negative. As $x \to 2$

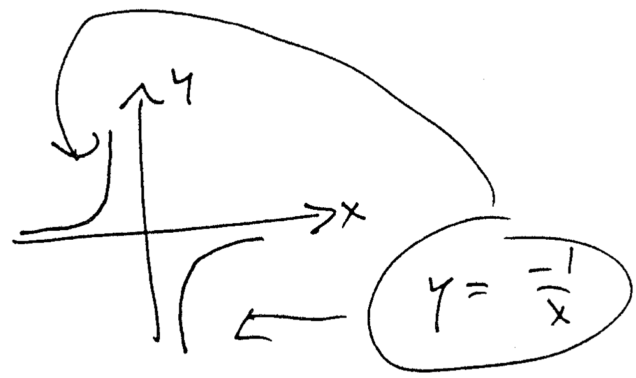
$2 - x \to 0$, so we are taking the limit of reciprocals of smaller & smaller negative numbers. These are large negative numbers. Hence

$$\lim_{\substack{x \to 2 \\ x > 2}} \frac{1}{2-x} = -\infty .$$

Note: The graph of $y = \frac{1}{x}$ looks like



So the graph of $y = -\frac{1}{x} = \frac{1}{-x}$ looks like

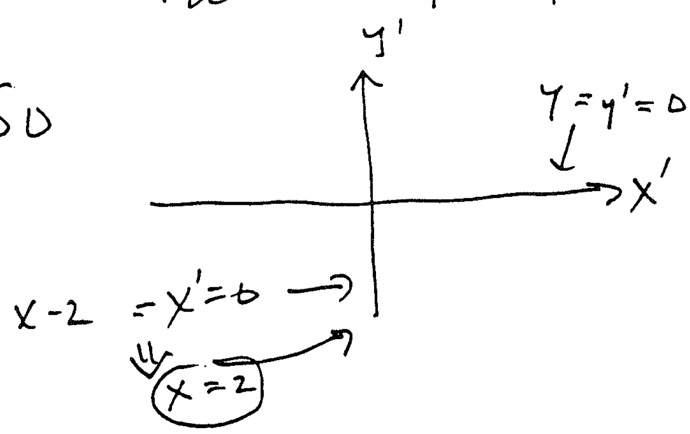


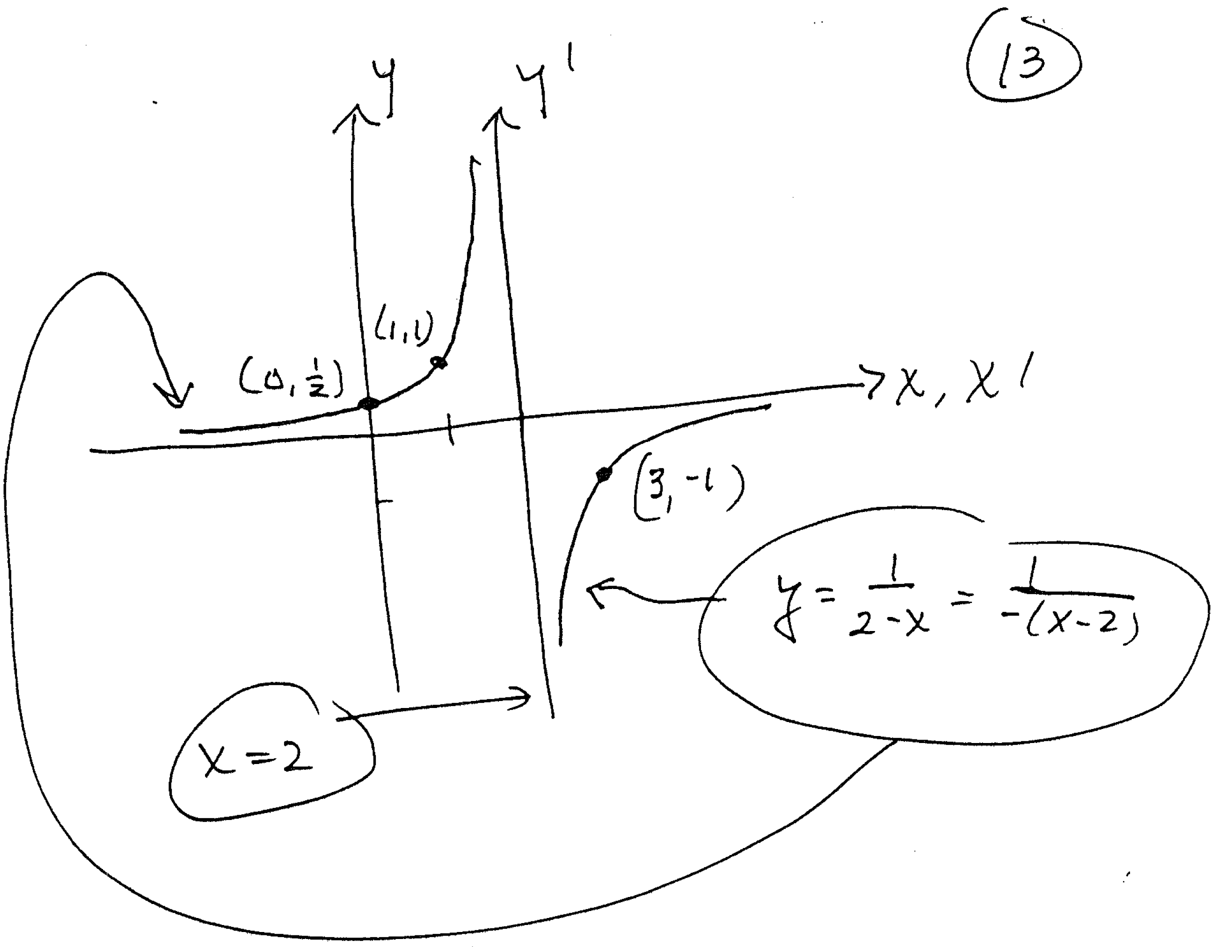
So the graph of $y' = \frac{1}{-x'}$ looks like

Now $y' = y = \frac{1}{2-x} = \frac{1}{-(x-2)} = \frac{1}{-x'}$

with $y' = y$ & $x' = x-2$

So





So clearly

$$\lim_{x \to 2^+} \frac{1}{2-x} = \lim_{\substack{x \to 2 \\ x > 2}} \frac{1}{-(x-2)} = -\infty$$

(i) $$\lim_{x \to 3} \sqrt{\frac{x^2-9}{x-3}} = \lim_{x \to 3} \sqrt{\frac{(x-3)(x+3)}{x-3}}$$

$$= \lim_{x \to 3} \sqrt{x+3}$$

$$= \sqrt{\lim_{x \to 3} (x+3)}$$

$$= \sqrt{3+3} = \sqrt{6}$$

(j) $$\lim_{x \to 0} \frac{\tan 5x}{x} = \lim_{x \to 0} 5 \frac{\tan(5x)}{5x}$$

$$= 5 \cdot \lim_{x \to 0} \frac{\tan(5x)}{5x}$$

$$= 5 \cdot 1 = 5$$

Since

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{\cos x} \cdot \frac{1}{x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$$= 1 \cdot \frac{1}{\cos(0)} = 1 \cdot \frac{1}{1}$$

$$= 1 \cdot 1 = 1$$

<u>Note</u>: $\tan(5x) \neq 5 \tan x$

So if you write

$$\lim_{x \to 0} \frac{\tan 5x}{x} = \lim_{x \to 0} \frac{5 \tan x}{x} = 5 \cdot 1 = 5$$

you will get <u>NO</u> <u>CREDIT!!!</u>