MATH 2107-601 **EXAM #1** NAME: KEY
(Print)

MONDAY

NAME: ______________
(Signature)

OCTOBER 4, 2004

[6:00 p.m. - 7:15 p.m.] STUDENT I. D.: ______________

INSTRUCTIONS: There are 2 parts to this exam, the main part plus an **optional** extra credit part (on the last page). The main part is worth 100 points while the extra credit part is worth 12 points - but any score over 100 will be truncated to 100.

Clarity of exposition is an integral part of a correct solution to any problem.

Good Luck.

PLEASE SIGN THE FOLLOWING STATEMENT:
On my honor, I declare that the work that follows is entirely my own. With regard to all the questions on this exam, I have neither given nor received help from anyone [including myself, say, via any type of cheat sheet or device (e.g., cell phone)]. Nor have I used a calculator of any sort (i.e., regular or programmable).

NAME: ______________
(Signature)

PLEASE DO NOT WRITE BELOW THIS LINE:

---

16 {
1. a. 3
b. 4
c. 6
d. 3

4 {
2. a. 1
b. 3

11 {
3. a. 2
b. 1
c. 4
d. 4

4 { 4. 4

Total: 35

10 { 5. 10

55 {
6. a. 5
b. 5
c. 5
d. 5
e. 5
f. 5
g. 5
h. 5
i. 5
j. 5
k. 5

Total: 65

EXTRA-CREDIT:
1. 5
2. 7

TOTALS:
Column 3: 12
Column 2: 65
Column 1: 35
Grand Total: 112 ↓ 100
Grade: A+

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1. (a) By definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

[3 points]

(b) Geometrically, $f'(x)$ is equal to the slope of the straight line tangent to the graph of f at the point $(x, f(x))$.

[4 points]

(c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and that $a \in \mathbb{R}$. Then, by definition,

$$\lim_{x \to a} f(x) = L$$

if and only if for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that

[6 points] $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$.

(d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and that $a \in \mathbb{R}$. Then, by definition, f is <u>continuous at</u> a if and only if

[3 points] (1) f is defined at a,

(2) $\lim_{x \to a} f(x)$ exists, and

(3) $\lim_{x \to a} f(x) = f(a)$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function. What characteristic of the graph of f enables you to tell at a glance

(a) that f is everywhere continuous?

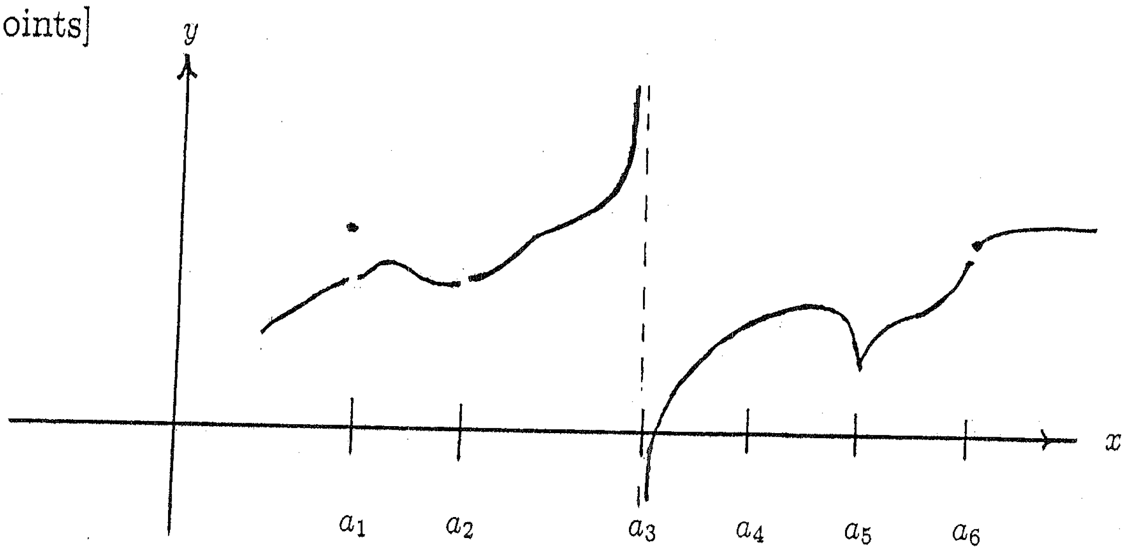
[1 point] The graph of f is unbroken.

(b) that f is everywhere differentiable?

[3 points] The graph of f is unbroken and has no corners, vertical tangents, or wild oscillations.

3. With reference to the graph below,

[11 points]



list all points p from the set $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ for which it is true that

(a) f is continuous at p. ANSWER: a_4, a_5

(b) f is differentiable at p. ANSWER: a_4

(c) $\lim_{x\to p} f(x)$ exists. ANSWER: a_1, a_2, a_4, a_5

(d) f is defined at p. ANSWER: a_1, a_4, a_5, a_6

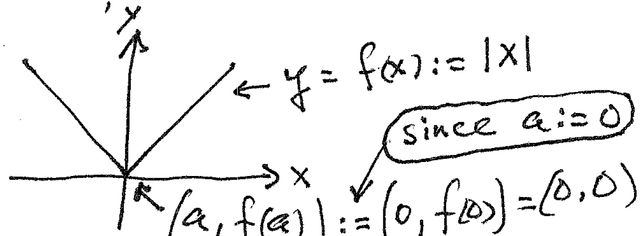
NOTE: Some points may possibly belong to more than one category. BUT, *Nota Bene*, the number of wrong answers will be subtracted from the number of right answers. This is to discourage "padded answers."

4. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a point $a \in \mathbb{R}$ such that f is continuous at a but $f'(a)$ does not exist.

[To get credit, you must give at least some indication of why f is continuous at a and why $f'(a)$ does not exist.]

[4 points] Define $f:\mathbb{R}\to\mathbb{R}$ by setting $f(x) := \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$ for each $x \in \mathbb{R}$, and let $a := 0 \in \mathbb{R}$.

Picture:



Then, $f:\mathbb{R}\to\mathbb{R}$ is everywhere continuous (&, in particular, at $x := a := 0$) because its graph is unbroken. Yet, $f'(0)$ does not exist because were it to exist $f'(0)$ would equal the slope of the straight line tangent to the graph of f at the point $(0, f(0)) := (0,0)$. But the graph of f has a corner at $(0, f(0))$ & hence can't have a tangent there!

5. Compute $f'(x)$ directly from the definition in case

$$f(x) = \frac{1}{4x+6}\,.$$

NOTE: No credit will be given for just the answer.

[10 points]

$$f'(x) = \underset{h\to 0}{L} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h\to 0}{L} \frac{\dfrac{1}{4(x+h)+6} - \dfrac{1}{4x+6}}{h}$$

$$= \underset{h\to 0}{L} \frac{\frac{1}{h}}{\frac{1}{h}} \cdot \frac{\dfrac{1}{4(x+h)+6}\cdot\dfrac{4x+6}{4x+6} - \dfrac{4(x+h)+6}{4(x+h)+6}\cdot\dfrac{1}{4x+6}}{h}$$

$$= \underset{h\to 0}{L} \frac{1}{h} \cdot \frac{4x+6 - 4(x+h)+6}{[4(x+h)+6]\cdot[4x+6]}$$

$$= \underset{h\to 0}{L} \frac{1}{h} \cdot \frac{4x+6 - [4x+4h+6]}{[4x+4h+6]\cdot[4x+6]}$$

$$= \underset{h\to 0}{L} \frac{1}{h} \cdot \frac{4x+6 - 4x - 4h - 6}{[4x+4h+6]\cdot[4x+6]}$$

$$= \underset{h\to 0}{L} \frac{1}{h} \cdot \frac{-4h}{[4x+4h+6]\cdot[4x+6]}$$

$$= \underset{h\to 0}{L} \frac{-4}{(4x+4h+6)\cdot(4x+6)}$$

$$= \frac{-4}{(4x+0+6)\cdot(4x+6)} = \frac{-4}{(4x+6)^2}$$

$$= \frac{-4}{[2(2x+3)]^2} = \frac{-4}{4(2x+3)^2} = \frac{-1}{(2x+3)^2}$$

6. Evaluate the following limits. **[No work, no credit.]**

[55 points: 5 each]

(a) If $f(x) = |x-7|$, then

$$\lim_{h\to 0^-} \frac{f(7)-f(7+h)}{h} = \underset{\substack{h\to 0\\ h<0}}{L} \frac{|7-7|+|7+h-7|}{h} = \underset{\substack{h\to 0\\ h<0}}{L} \frac{-|h|}{h}$$

$$= \underset{\substack{h\to 0\\ h<0}}{L} \frac{-(-h)}{h} = \underset{\substack{h\to 0\\ h<0}}{L} \frac{h}{h} = \underset{\substack{h\to 0\\ h<0}}{L} 1 = 1$$

(b)

$$\lim_{x\to 2} \frac{x^4-16}{x^5-32} = \underset{x\to a}{L} \frac{x^4-a^4}{x^5-a^5} = \underset{x\to a}{L} \frac{(x-a)\cdot(x^3\cdot a^0 + x^2\cdot a^1 + x^1\cdot a^2 + x^0\cdot a^3)}{(x-a)\cdot(x^4\cdot a^0 + x^3\cdot a^1 + x^2\cdot a^2 + x^1\cdot a^3 + x^0\cdot a^4)}$$

$\boxed{a=2}$

$$= \underset{x\to a}{L} \frac{x^3 + x^2\cdot a + x\cdot a^2 + a^3}{x^4 + x^3\cdot a + x^2\cdot a^2 + x\cdot a^3 + a^4}$$

$\boxed{a=2}$

$$= \underset{x\to a}{L} \frac{a^3 + a^2\cdot a + a\cdot a^2 + a^3}{a^4 + a^3\cdot a + a^2\cdot a^2 + a\cdot a^3 + a^4} = \frac{4a^3}{5\cdot a^4} = \frac{4}{5\cdot a} = \frac{4}{5\cdot 2} = \frac{2\cdot 2}{5\cdot 2} = \frac{2}{5}$$

(c)

$$\lim_{x\to 4} \frac{x^2+4x-32}{x^2-6x+8} = \underset{x\to 4}{L} \frac{(x-4)\cdot(x+8)}{(x-4)\cdot(x-2)} = \underset{x\to 4}{L} \frac{x+8}{x-2}$$

$$= \frac{4+8}{4-2} = \frac{12}{2} = 6$$

$$\begin{array}{r|rrr} 4 & 1 & 4 & -32 \\ & & 4 & 32 \\ \hline & 1 & 8 & \boxed{0} \end{array}$$

$$\begin{array}{r|rrr} 4 & 1 & -6 & 8 \\ & & 4 & -8 \\ \hline & 1 & -2 & \boxed{0} \end{array}$$

(d)

$$\lim_{x\to 0} \frac{\tan 9x}{\sin 3x} = \underset{x\to 0}{L}\ 9\cdot\frac{\tan 9x}{9x}\cdot\frac{3x}{\sin 3x}\cdot\frac{1}{3}$$

$$= 9\cdot 1\cdot 1\cdot\frac{1}{3} = \frac{9}{3} = 3$$

(e)

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \underset{x\to 0}{L} \frac{\cos x - 1}{x^2}\cdot\frac{\cos x + 1}{\cos x + 1} = \underset{x\to 0}{L} \frac{\cos^2 x - 1}{x^2\cdot(\cos x + 1)}$$

$$\boxed{\begin{array}{c} s^2 + c^2 = 1 \\ \Downarrow \\ c^2 - 1 = -s^2 \end{array}}$$

$$= \underset{x\to 0}{L} \frac{-\sin^2 x}{x^2\cdot(\cos x + 1)} = \underset{x\to 0}{L} -\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{\cos x + 1}$$

$$= -1\cdot 1^2\cdot\left(\frac{1}{1+1}\right) = \frac{-1}{2}$$

(f)

$$\lim_{x\to 0} \frac{\sec x - 1}{x} = \underset{x\to 0}{L} \frac{\sec x - 1}{x}\cdot\frac{\sec x + 1}{\sec x + 1} = \underset{x\to 0}{L} \frac{\sec^2 x - 1}{x\cdot(\sec x + 1)}$$

$$\boxed{\frac{s^2+c^2}{c^2} = \frac{1}{c^2} = \left(\frac{1}{c}\right)^2 = \sec^2}$$

$$\frac{s^2}{c^2} + \frac{c^2}{c^2}$$

$$t^2 + 1$$

$$\Downarrow\ \boxed{\sec^2 x - 1 = \tan^2 x}$$

$$= \underset{x\to 0}{L} \frac{\tan^2 x}{x}\cdot\frac{1}{\sec x + 1} = \underset{x\to 0}{L} \frac{\tan x}{x}\cdot\frac{\tan x}{\sec x + 1}$$

$\boxed{\text{OR}}$

$$\underset{x\to 0}{L} \frac{\tan^2 x}{x^2}\cdot\frac{x}{\sec x + 1} = 1\left(\frac{0}{2}\right) = 0$$

$$\left(= 1\cdot\left(\frac{0}{1+1}\right) = 1\cdot\left(\frac{0}{2}\right) = 1\cdot 0 = 0\right.$$

6. (Continued): Evaluate the following limits. **[No work, no credit.]**

(g) $\lim_{x\to-\infty}\frac{-18x}{\sqrt{81x^2+1}} = \underset{X\to-\infty}{L}\frac{-18X}{\sqrt{X^2\cdot\left(81+\frac{1}{X^2}\right)}} = \underset{X\to-\infty}{L}\frac{-18X}{\sqrt{X^2}\cdot\sqrt{81+\frac{1}{X^2}}}$

$= \underset{X\to-\infty}{L}\frac{-18X}{|X|\cdot\sqrt{81+\frac{1}{X^2}}} = \underset{X\to-\infty}{L}\frac{-18X}{-X\cdot\sqrt{81+\frac{1}{X^2}}}$

$= \underset{X\to-\infty}{L}\frac{18}{\sqrt{81+\frac{1}{X^2}}} = \frac{18}{\sqrt{81+0}} = \frac{18}{9} = 2$

(h) $\lim_{x\to5}\frac{5-x^2}{25-x} = \frac{5-25}{25-5} = \frac{-(25-5)}{25-5} = -1$

OR $\longrightarrow \overset{\shortparallel}{\frac{-20}{20}} = -1$

(i) $\lim_{x\to-\infty}\frac{3x}{x-3} = \underset{X\to-\infty}{L}\frac{3X}{X-3}\cdot\frac{\frac{1}{X}}{\frac{1}{X}} = \underset{X\to-\infty}{L}\frac{3}{1-\frac{3}{X}}$

$= \frac{3}{1-0} = \frac{3}{1} = 3$

(j) $\lim_{x\to3^-}\frac{3x}{x-3} = \underset{\substack{X\to3\\X<3}}{L}\frac{3X}{X-3} = \frac{9}{0^-} = -\infty.$

The Value of $X-3$: $\underset{3}{----0+++++}$ $\Rightarrow$ For $X<3$, $X-3$ = smaller - bigger = negative.

But this negative number goes to 0 as X approaches 3 from the Left.

(k) $\lim_{x\to11}\frac{9-\sqrt{x+70}}{11-x} = \underset{X\to11}{L}\frac{9-\sqrt{X+70}}{11-X}\cdot\frac{9+\sqrt{X+70}}{9+\sqrt{X+70}}$

$= \underset{X\to11}{L}\frac{81-(X+70)}{(11-X)\cdot(9+\sqrt{X+70})} = \underset{X\to11}{L}\frac{81-X-70}{(X-11)\cdot(9+\sqrt{X+70})}$

$= \underset{X\to11}{L}\frac{11-X}{(11-X)\cdot(9+\sqrt{X+70})} = \underset{X\to11}{L}\frac{1}{9+\sqrt{X+70}}$

$= \frac{1}{9+\sqrt{11+70}} = \frac{1}{9+\sqrt{81}} = \frac{1}{9+9} = \frac{1}{18}$

6. (Continued): Evaluate the following limits. **[No work, no credit.]**

(g) $\displaystyle\lim_{x\to-\infty}\frac{-18x}{\sqrt{81x^2+1}} =$

(h) $\displaystyle\lim_{x\to 5}\frac{5-x^2}{25-x} =$

(i) $\displaystyle\lim_{x\to-\infty}\frac{3x}{x-3} =$

(j) $\displaystyle\lim_{x\to 3^-}\frac{3x}{x-3} = \underset{\substack{X\to3\\X<3}}{L}\ 3\left[\frac{X-3+3}{X-3}\right] = \underset{\substack{X\to3\\X<3}}{L}\ 3\left[\frac{X-3}{X-3}+\frac{3}{X-3}\right]$

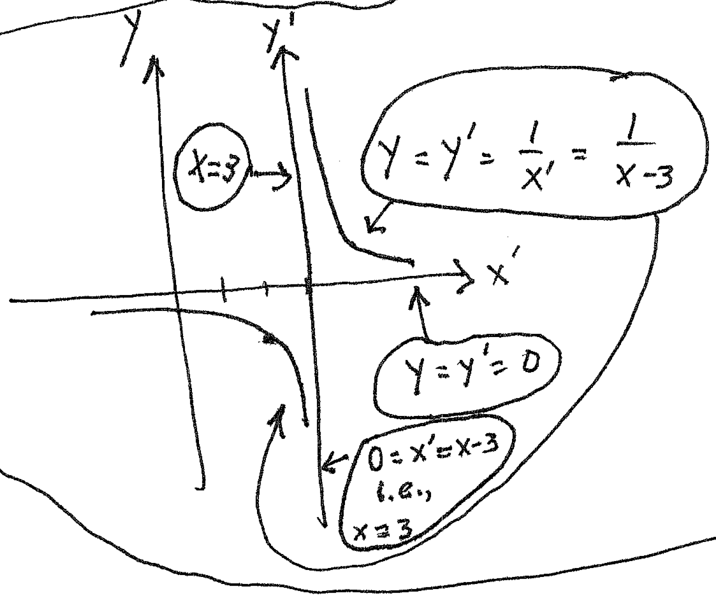
$\displaystyle = \underset{\substack{X\to3\\X<3}}{L}\ 3\left[1+\frac{3}{X-3}\right] = \underset{\substack{X\to3\\X<3}}{L}\left(3+\frac{9}{X-3}\right)$

$\displaystyle = \underset{\substack{X\to3\\X<3}}{L}\left[3+9\cdot\left(\frac{1}{X-3}\right)\right]$

$\text{"="}\ 3 + -\infty$

$= -\infty$

(k) $\displaystyle\lim_{x\to 11}\frac{9-\sqrt{x+70}}{11-x} =$



EXTRA-CREDIT: [**12 points**: **But**, **any score over 100 will be truncated to 100**.]

1. Evaluate the following limits. [**1 point for each correct answer; but you must show your work to get credit!**]

[5 points: 1 each]

(a) $\lim_{x\to1}\frac{\tan(x^4-1)}{x-1} = \underset{x\to1}{L}\frac{\tan(x^4-1)}{x-1}\cdot\frac{x^3+x^2+x+1}{(x^3\cdot1^0+x^2\cdot1^1+x^1\cdot1^2+x^0\cdot1^3)}$

$= \underset{x\to1}{L}\frac{\tan(x^4-1)}{x^4-1}\cdot(x^3+x^2+x+1) = 1\cdot(1^3+1^2+1+1)$

$= 1\cdot(1+1+1+1)$

$= 1\cdot4 = 4$

$$\begin{array}{l} x^2-2x+3 \\ x+1 \\ \hline x^3-2x^2+3x \\ \quad\; x^2-2x+3 \\ \hline x^3-x^2+x+3 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 1 & 3 \\ & & -1 & 2 & -3 \\ \hline & 1 & -2 & 3 & \boxed{0} \end{array}$$

(b) $\lim_{x\to-1}\frac{x^3-x^2+x+3}{x+1} = \underset{x\to-1}{L}\frac{[x-(-1)][x^2-2x+3]}{x-(-1)} = \underset{x\to-1}{L}(x^2-2x+3)$

$= (-1)^2-2(-1)+3 = 1+2+3 = 6$

(c) $\lim_{x\to1^-}\frac{x^3-x^2+x+3}{x+1} = \underset{\substack{x\to1\\x<1}}{L}\frac{1^3-1^2+1+3}{1+1} = \frac{1-1+1+3}{2} = \frac{4}{2} = 2$

OR $\underset{\substack{x\to1\\x<1}}{L}(x^2-2x+3) = 1^2-2\cdot1+3 = 1-2+3 = 4-2 = 2$

(d) $\lim_{x\to0}\frac{\sin6x\cdot\sin4x}{x} = \underset{x\to0}{L}\,6\cdot\frac{\sin6x}{6x}\cdot\sin4x = 6\cdot1\cdot0 = 0$

(e) $\lim_{x\to0}\frac{\sin6x\cdot\sin4x}{x^2} = \underset{x\to0}{L}\,6\cdot\frac{\sin6x}{6x}\cdot\frac{\sin4x}{4x}\cdot4$

$= 6\cdot1\cdot1\cdot4 = 24$

2. Find the equation of the oblique (= slant) asymptote to the graph of

Check:
$$\begin{array}{l} 4x+2 \\ x^2+1 \\ \hline 4x^3+2x^2+4x+2 \\ \quad +\;2x+6 \\ \hline 4x^3+2x^2+6x+8 \;✓ \end{array}$$

$$y = f(x) = \frac{4x^3+2x^2+6x+8}{x^2+1} = \underbrace{4x+2}_{:=\,g(x)} + \underbrace{\frac{2x+6}{x^2+1}}_{h(x)}$$

$= \frac{(4x+2)(x^2+1)+2x+6}{x^2+1}$

[7 points]

SOLUTION:

$$\begin{array}{r|l} & 4x+2 \\ \hline x^2+1 & 4x^3+2x^2+6x+8 \\ & 4x^3 \qquad\quad +4x \\ \hline & \qquad 2x^2+2x+8 \\ & \qquad 2x^2 \qquad +2 \\ \hline & \qquad\qquad 2x+6 \end{array}$$

Hence, $\underset{x\to\pm\infty}{L}f(x)-g(x) = \underset{x\to\pm\infty}{L}h(x) = \underset{x\to\pm\infty}{L}\frac{2x+6}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \underset{x\to\pm\infty}{L}\frac{\frac{2}{x}+\frac{6}{x^2}}{1+\frac{1}{x^2}}$

$= \frac{\pm0+0}{1+0} = \frac{0}{1} = 0 \Rightarrow$ $y = 4x+2$ is the slant asymptote to the graph of f