#1. A pebble is dropped from a height of 600 feet. Find the pebble's velocity when it hits the ground.
[Use $s(t) = -16t^2 + v_0 t + s_0$]

$$V_0 = 0$$

$$S_0 = 600$$

$$s(t) = -16t^2 + V_0 t + S_0$$

$$= -16t^2 + 0 \cdot t + 600$$

$$\therefore \quad s(t) = -16t^2 + 600$$

$$0 = s(t) = -16t^2 + 600$$

$$16t^2 = 600$$

$$t^2 = \frac{600}{16} = 37.5$$

$$t = \sqrt{37.5} \approx 6.123724357$$

$$S(t) = -16t^2 + 600$$

$$\Downarrow$$

$$S'(t) = -32t$$

$$\therefore \quad s'(t)\Big|_{t=\sqrt{37.5}} = S'(\sqrt{37.5}) = -32\sqrt{37.5}$$

$$= (-32)(6.123724357)$$

$$= 195.9591794 \text{ ft/sec}$$

Therefore

$\boxed{s'(t) \approx 195.96 \text{ feet/sec}}$

#2. A ball was designed to have a radius of 2 inches. However, when the ball was manufactured, its radius was too short by 1/10 of an inch. Use differentials to approximate the resulting error in volume.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

$$\therefore \frac{dV}{dr} = 4\pi r^2$$

so $dV = 4\pi r^2\, dr$

$$= 4\pi \cdot (2 \text{ inches})^2 \cdot (-.1 \text{ inch})$$

$$= 4\pi \cdot 4\,(-.1)\,(\text{inches})^3$$

$$= -1.6\pi \text{ cubic inches}$$

$$\approx -5.026\ 548\,246\ (\text{inches})^3$$

$$\approx -5.03 \text{ cubic inches}.$$

$\boxed{\#3.}$ An object dropped from the top of a brick building takes 10 seconds to hit the ground. How high is the building?
[use $a(t) = -32\ ft/sec^2$]

$$\frac{d\,V(t)}{dt} = a(t) = -32$$
$$\Downarrow$$
$$d\,V(t) = a(t)\,dt = -32\,dt$$
$$\Downarrow$$
$$\frac{d\,s(t)}{dt} = V(t) = \int d\,V(t) = \int a(t)\,dt = \int -32\,dt$$
$$= -32t + V_0$$

$$\frac{ds(t)}{dt} = -32t + V_0$$
$$\Downarrow$$
$$ds(t) = (-32t + V_0)\,dt$$
$$\Downarrow$$
$$s(t) = \int d\,s(t) = \int (-32t + V_0)\,dt$$
$$= -32\frac{t^2}{2} + V_0 t + S_0 = -16t^2 + S_0$$
$$\because V_0 = 0$$

$$0 = s(10) = -16(10)^2 + S_0 \Rightarrow \boxed{S_0 = 1600}$$

<u>Note:</u> Compare this problem with problem # 43 on page 268 of the Text Calculus by Larson / Hostetler / Edwards 4th Edition.

43 An object is dropped from a balloon that is stationary at 1600 feet above the ground. (a) Express its height above the ground as a function of t. (b) How long does it take the object to reach the ground?

(a)

$$\frac{d\,v(t)}{dt} = a(t) = -32\,ft/sec^2$$

$$\frac{d\,s(t)}{dt} = v(t) = \int d\,v(t) = \int a(t)\,dt = \int -32\,dt$$

$$= -32t + v_0$$

$$s(t) = \int d\,s(t) = \int v(t)\,dt = \int (-32t + v_0)\,dt$$

$$= \int -32\,t + v_0\,dt$$

$$\therefore\ s(t) = -32\frac{t^2}{2} + v_0 t + s_0$$

$$= -16t^2 + v_0 t + s_0$$

But $v_0 = 0$ & $s_0 = 1600$

so $\boxed{s(t) = -16t^2 + 1600}$

(b) $0 = s(t) = -16t^2 + 1600$

$$\Downarrow$$

$$16t^2 = 1600$$

$$\Downarrow$$

$$t^2 = \frac{1600}{16} = 100$$

$$\Downarrow$$

$$t = \pm 10$$

But $t = -10$ has no physical significance

$\therefore \quad t = 10$

Check: $s(t) = -16t^2 + 1600$

So $s(10) = -16 \cdot 10^2 + 1600$

$= -16 \cdot 100 + 1600$

$= -1600 + 1600 = 0$.

#4. The radius of the earth is measured to be 4261 miles. If the measurement is correct to within 20 miles, estimate the possible error in the volume of the earth. [use $V = \frac{4}{3}\pi r^3$]

$$V(r) = \frac{4}{3}\pi r^3$$

$$\frac{d\,V(r)}{dr} = V'(r) = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

$$\Downarrow$$

$$d\,V(r) = 4\pi r^2\,dr$$

i.e.,

$$V = \frac{4}{3}\pi r^3$$

$$\Downarrow$$

$$\frac{dV}{dr} = \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2$$

$$\Downarrow$$

$$dV = 4\pi r^2\,dr$$

But $V'(r) = \lim\limits_{h \to 0} \dfrac{V(r+h) - V(r)}{h}$

So, for small h (relative to r)

$$\frac{V(r+h) - V(r)}{h} \approx V'(r)$$

"$\approx$" means: approximately equal to

Hence, for h small relative to r

$$V(r+h) - V(r) \approx V'(r)\cdot h$$

Upon setting $dr := h$, we find that the

$$\begin{aligned}\text{error} = V(r+h) - V(r) &\approx V'(r)\cdot h \\ &= V'(r)\cdot dr \\ &= \frac{dV}{dr}\cdot dr \\ &= dV \\ &= \text{the differential of } V\end{aligned}$$

Hence,

$$\begin{aligned}\text{the error} = dV = \frac{dV}{dr}\,dr &= 4\pi r^2\,dr \\ &= 4\pi r^2\cdot(\pm 20) \\ &= 4\pi\,(\underset{\text{feet}}{4261})^2\cdot(\underset{\text{feet}}{\pm 20}) \\ &= \pm\, 80\pi\cdot(4261)^2 \text{ cubic feet} \\ &= \pm\, 80\pi\cdot(18156121) \text{ cubic feet} \\ &= \pm\, 1{,}452{,}489{,}680\,\pi \text{ cubic feet} \\ &\approx \pm 4{,}563{,}130{,}909 \text{ cubic feet}^{*}\end{aligned}$$

* using the value $\pi \approx 3.141592654$

Note: As a check we calculate the actual volumes as follows:

$$V(r) = \frac{4}{3}\pi r^3$$

Now $\frac{4}{3}\pi \approx \frac{4}{3}(3.141592654) = \frac{12.566370\,62}{3}$

$= 4.188\ 790\ 207$

Hence

$$V(4281) = \frac{4\pi}{3}(4281)^3$$

$= \frac{4\pi}{3}(7.84577200 \times 10^{10})$

$= 3.28642929\,4 \times 10^{11}$

$= 328,642,929,400$

$$V(4261) = \frac{4\pi}{3}(4261)^3$$

$= \frac{4\pi}{3}(7.736\ 323\ 158 \times 10^{10})$

$= 3.240\ 583\ 468 \times 10^{11}$

$= 324,058,346,800$

$$V(4241) = \frac{4\pi}{3}(4241)^3$$

$= \frac{4\pi}{3}(7.627\,896\,952 \times 10^{10})$

$= 3.195\ 166\,005 \times 10^{11}$

$= 319,516,600,500$

Thus

$$V(4281) = 328{,}642{,}929{,}400 \text{ cubic feet}$$

$$V(4261) = 324{,}058{,}346{,}800 \text{ cubic feet}$$

$$V(4241) = 319{,}516{,}600{,}500 \text{ cubic feet}$$

Now

$$\begin{array}{r} 328{,}642{,}929{,}400 \\ -\ 324{,}058{,}346{,}800 \\ \hline 4{,}584{,}582{,}600 \end{array}$$

While

$$\begin{array}{r} 324{,}058{,}346{,}800 \\ -\ 319{,}516{,}600{,}500 \\ \hline 4{,}541{,}746{,}300 \end{array}$$

Thus

$$V(4281) = V(4261) + 4{,}584{,}582{,}600$$

$$V(4261) = V(4261) + 0$$

$$V(4241) = V(4261) - 4{,}541{,}746{,}300$$

By contrast, the estimated error was

$\pm$ 4,563,130,909 cubic feet.

$\boxed{\#5.}$ A rocket is shot straight up at a velocity of 300 feet/second from a platform 100 feet high. What will its height be t seconds after launch? [Assume this takes place on a planet on which gravity causes an acceleration of -20 feet/(second)^2.]

Given:

$$\left\{\begin{array}{l} \underbrace{S_0}_{\text{the initial distance}} := \underbrace{S(0)}_{\text{the distance at time } t=0} = 100 \text{ feet} \\[2ex] \underbrace{V_0}_{\text{the initial velocity}} = \underbrace{V(0)}_{\text{the velocity at time } t=0} = S'(0) = 300 \text{ feet/second} \\[2ex] a(t) = V'(t) = S''(t) = -20 \text{ feet/(sec)}^2. \end{array}\right.$$

$$\frac{d\,V(t)}{dt} = V'(t) = a(t) = -20$$

$$\Downarrow$$

$$dV(t) = a(t)\,dt = -20\,dt$$

$$\Downarrow$$

$$V(t) = \int dV(t) = \int a(t)\,dt = \int -20\,dt = -20t + V_0$$

i.e.,

$$\frac{d\,v(t)}{dt} = a(t) = -20$$

$$\Downarrow$$

$$\frac{d\,s(t)}{dt} = v(t) = \int d\,v(t) = \int a(t)\,dt = \int -20\,dt$$

$$= -20t + V_0$$

i.e.

$$\frac{d\,s(t)}{dt} = -20t + V_0$$

$$\Downarrow$$

$$d\,s(t) = (-20t + V_0)\,dt$$

$$\Downarrow$$

$$s(t) = \int d\,s(t) = \int (-20t + V_0)\,dt$$

$$= -20\frac{t^2}{2} + V_0 t + S_0$$

$$\therefore\ s(t) = -10t^2 + V_0 t + S_0$$

i.e., $\boxed{s(t) = -10t^2 + 300t + 100}$

Check:

1 $$S(t) = -10t^2 + 300t + 100$$

$$\Downarrow$$

$$\boxed{S_0 = S(0) = 100}$$ ✓

2 $$S(t) = -10t^2 + 300t + 100$$

$$\Downarrow$$

$$V(t) = S'(t) = -20t + 300$$

$$\Downarrow$$

$$\boxed{V_0 = V(0) = S'(0) = 300}$$ ✓

3 $$S(t) = -10t^2 + 300t + 100$$

$$\Downarrow$$

$$V(t) = S'(t) = -20t + 300$$

$$\Downarrow$$

$$\boxed{a(t) = V'(t) = S''(t) = -20}$$ ✓

$\boxed{\#6}$ Use differentials to get an approximation for $\sqrt[3]{996}$.

[Note: $\sqrt[3]{1000} = 10$]

Solution: Set $f(x) = \sqrt[3]{x} = x^{1/3}$

Then $$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$$

We want $f(a+h)$

for $a = 1000$ & $h = -4$

i.e., $$\sqrt[3]{996} = (996)^{1/3} = f(996)$$
$$= f(1000 - 4)$$
$$= f(1000 + -4)$$

Now, for h small relative to a

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

Since

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Now,

$$\frac{f(a+h) - f(a)}{h} \approx f'(a)$$

$$\Downarrow$$

$$f(a+h) - f(a) \approx f'(a) \cdot h$$

$$\Downarrow$$

$$\boxed{f(a+h) \approx f(a) + f'(a) \cdot h}$$

Here:

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$a = 1000$$

$$h = -4$$

Then $f(a+h) = f(1000 + -4) = f(996)$

$= \sqrt[3]{996}$

while

$f(a) = f(1000) = \sqrt[3]{1000} = 10$

$f'(a) = \frac{1}{3\, a^{2/3}} = \frac{1}{3\,(1000)^{2/3}}$

$= \frac{1}{3 \cdot (10)^2} = \frac{1}{3 \cdot 100} = \frac{1}{300}$

$f'(a) \cdot h = \frac{1}{300}(-4) = \frac{-4}{300} = -.01\overline{3}$

Thus

$\sqrt[3]{996} = f(a+h) \approx f(a) + f'(a) \cdot h$

$= 10 + \frac{1}{3 \cdot 100}(-4)$

$= 10 - .01\overline{3}$

$= 9.98\overline{6} = 9.986666\ldots$

In fact, $\sqrt[3]{996} \simeq 9.98664849$

$f(x) = x^{1/3}$

<u>Note:</u>

$$\sqrt[3]{996} = (996)^{1/3} \stackrel{\swarrow}{=} f(996)$$

$$= f(1000-4)$$

$$= f(1000 + -4)$$

$$= f(a+h)$$

$$\approx f(a) + f'(a) \cdot h$$

$$= a^{1/3} + \frac{1}{3} a^{-2/3} \cdot h$$

$$= (1000)^{1/3} + \frac{1}{3\,(1000)^{2/3}} \cdot (-4)$$

$$= 10 + \frac{1}{3 \cdot (10)^2} (-4)$$

$$= 10 + \frac{-1.3333\ldots}{100}$$

$$= 10 - .013333\ldots = 9.98666\ldots$$

$$= 9.98\overline{6}$$

Differentials

Def: Let a & b be real numbers with $a < b$. Set

$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ = the closed interval from a to b

&

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ = the open interval from a to b.

Def. Let $f:[a,b] \to \mathbb{R}$ be a function which is defined & continuous on the closed interval $[a,b]$ & differentiable on the open interval (a,b) & let $x \in (a,b)$, i.e., suppose $a < x < b$.

With x we define a new independent variable called the differential of x & denoted dx. Note: "dx" is a symbol. It does not signify "d" times "x" any more than "Δx" signifies "Δ" times "x". Recall, "Δx" stood for the difference in the x-values.

Note: dx can be any real number.

We then define the differential of y of the dependent variable y via

$$dy := f'(x) \cdot dx$$

In short, given f a differentiable function of x & any point x of an open interval on which f is defined, we let

$$dx := \text{any real number}$$

&

$$dy := f'(x) \cdot dx$$

It follows that if $dx \neq 0$, then

$$\frac{dy}{dx} = f'(x)$$

i.e,

$$\frac{dy}{dx} = f'(x) = \frac{dy}{dx}$$

Thus, when $dx \neq 0$, the ratio of dy to dx equals the derivative of y with respect to x.

This has nice consequences for the chain rule.

For recall that if $y = f(u)$ is a differentiable function of u & if $u = g(x)$ is a differentiable function of x, then

$$y = f(u) = f[g(x)] = f \circ g(x)$$

is a differentiable function of x

&

$$\frac{dy}{dx} = (f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

$$= f'(u) \cdot g'(x)$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}$$

Now prior to the introduction of differentials it makes no sense to look at

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

& say "of course" $\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$

because

$$\frac{dy}{du} = \frac{d\,y}{du} = \frac{d}{du}(y)$$

&

$$\frac{du}{dx} = \frac{d\,u}{dx} = \frac{d}{dx}(u)$$

& so cancelling the du's & saying

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx} = \frac{dy}{dx}$$

makes about as much sense as cancelling the F's in

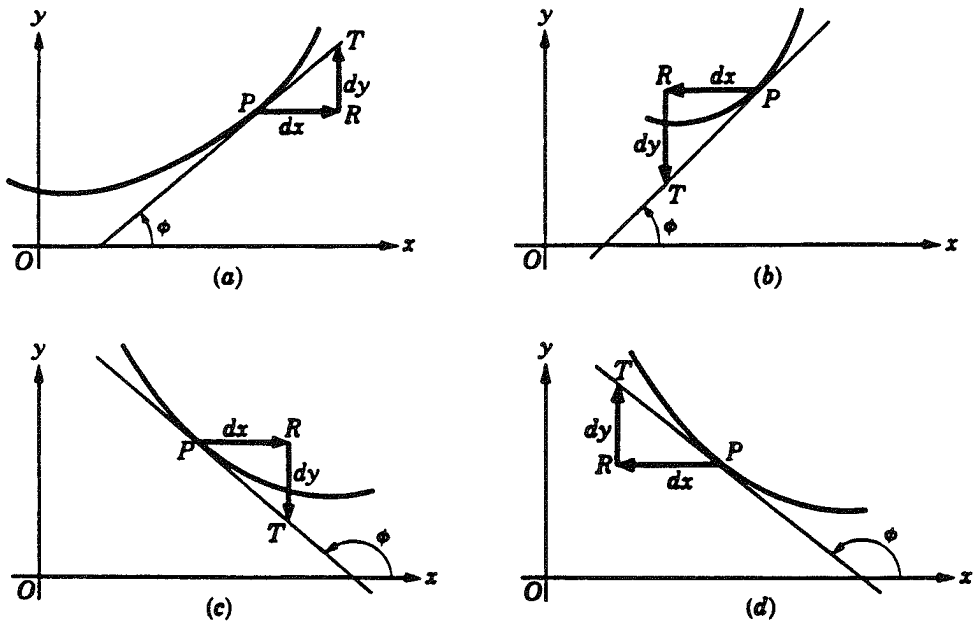
$\frac{E}{F}$ & concluding that

$$\frac{E}{F} = \frac{\mathbf{E}}{\mathbf{F}} = \frac{\cancel{\mathbf{E}}}{\cancel{\mathbf{F}}} = -$$

But once we let dx = any real number & let $dy = f'(x) \cdot dx$, then if $dx \neq 0$

$$\frac{dy}{dx} = \frac{f'(x) \cdot \cancel{dx}}{\cancel{dx}} = f'(x).$$

Picture



This entire discussion can be made rigorous by introducing the notion of the tangent bundle to the graph of f. Roughly speaking one proceeds as follows.

1st we let

$$\mathbb{R}^2 := \mathbb{R} \times \mathbb{R} := \{(x,y) \mid x \in \mathbb{R} \ \& \ y \in \mathbb{R}\}$$

$$= \text{the plane.}$$

Next, we let

$$X : \mathbb{R}^2 \to \mathbb{R}$$

&

$$Y : \mathbb{R}^2 \to \mathbb{R}$$

be the coordinate functions:

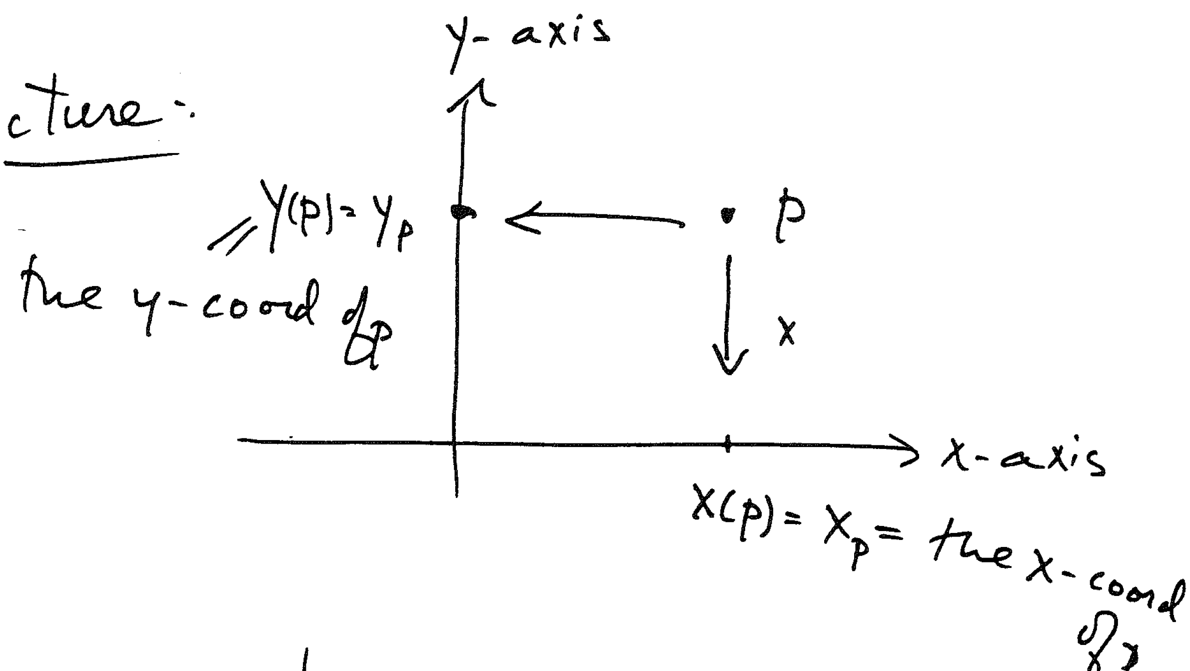
For any point p in the plane (i.e., for any $p \in \mathbb{R}^2$), we let

$$x_p := X(p) = \text{the } x\text{-coordinate of } p$$

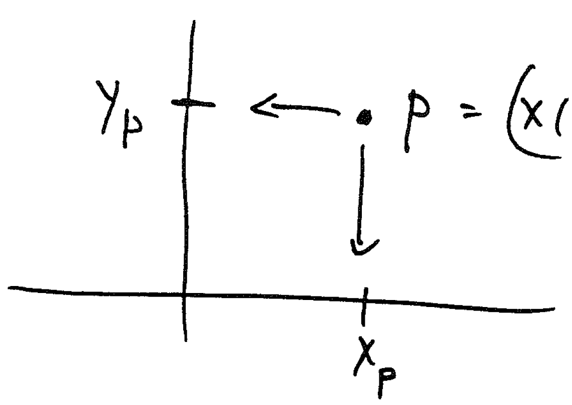
&

$$y_p := Y(p) = \text{the } y\text{-coordinate of } p$$

<u>Picture</u>:



Then



$$p = \big(X(p), Y(p)\big) = (x_p, y_p)$$

$\shortparallel$ the x-y coordinates of p.

Thus, for any point $p \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ = the plane

$$p = (x(p), y(p)) = (x_p, y_p) = \text{the } x\text{-}y \text{ coordinates of } p$$

Now let

$$C := \text{the graph of } f$$
$$:= \{ (x, f(x)) \mid a \le x \le b \}$$

["C" stands for "curve"!]

For each point $p \in C$ = the graph of f

let

$T_p :=$ the straight line tangent to the graph of f at the point p.

<u>Note</u>: If $p = (x, f(x))$ then, as we know, the slope of T_p is $f'(x)$.

Next, for each point $p \in C$ = the graph of f

we shall define functions

$$dx(p) := dx_p : T_p \to \mathbb{R}$$

& $$dy(p) := dy_p : T_p \to \mathbb{R}$$

<u>Note</u>: To say that

$$dx_p : T_p \to \mathbb{R}$$

is a function is to say that dx_p assigns to each point q of the tangent line, T_p, a real number.

Similar remarks hold for dy_p.

<u>Here are the definitions:</u>

Define functions

$$dx(p) := dx_p : T_p \to \mathbb{R}$$

& $$dy(p) := dy_p : T_p \to \mathbb{R}$$

as follows: For each point $q \in T_p$

$$dx_p(q) := \text{the directed } x\text{-distance from } x(p) := x_p \text{ to } x(q) := x_q$$

$$:= x(q) - x(p) := x_q - x_p$$

the x-coordinate of p

the x-coord. of q

& similarly

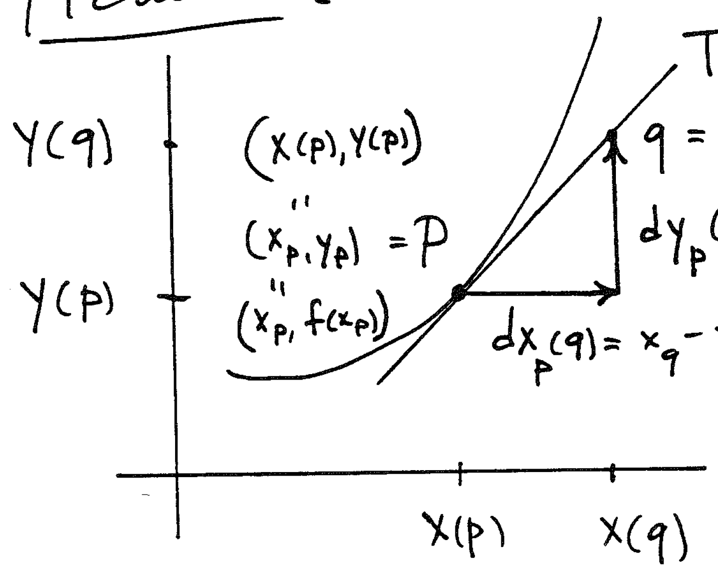
$dy_p(q)$:= the directed y-distance from $y(p)$ to $y(q)$

$:= y(q) - y(p) = y_q - y_p$

the y-coordinate of q

the y-coord. of p

Picture (where $q \neq p$)



Since the slope of T_p is $f'(x_p)$

we have that

$$\frac{dy_p(q)}{dx_p(q)} = f'(x_p)$$

i.e.,

$$\frac{dy_p(q)}{dx_p(q)} = \frac{y(q)-y(p)}{x(q)-x(p)} = \frac{y_q - y_p}{x_q - x_p} = f'(x_p)$$

$$= f'[x(p)]$$

i.e.,

$$dy_p(q) = f'[x(p)] \cdot dx_p(q)$$

This functional equation holds for all points q in the tangent line T_p (i.e., for all $q \in T_p$) & so we have the identity

$$dy(p) := dy_p = f'[x(p)] \cdot dx_p := f'[x(p)] \cdot dx(p)$$

i.e. $$dy(p) = f'[x(p)] \cdot dx(p)$$

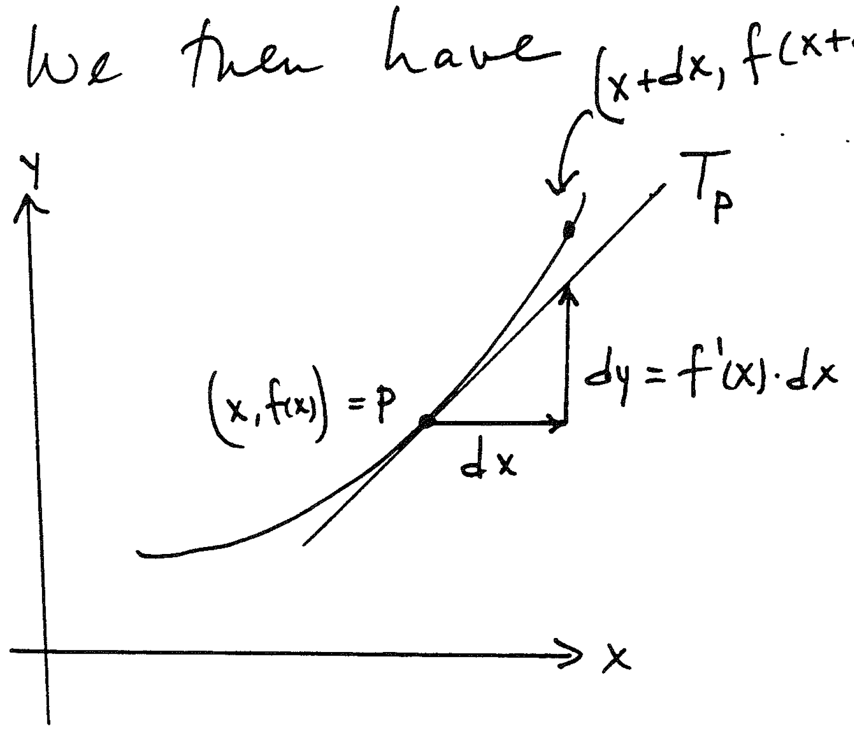
as functions. This equation is in turn an identity on the tangent line T_p & so

$$dy = f'(x) \cdot dx$$

(as functions!).

In this spirit, we redraw the previous picture omitting some of the subscripts:

We then have



Thus, if dx denotes the change in x, then dy = the change in y along the tangent line [to the graph of f at the point $(x, f(x))$] produced by a change of dx units in x.

The actual change in y would be given by

$$f(x+dx) - f(x)$$

The error is $f(x+dx) - [f(x) + dy]$

$$= f(x+dx) - f(x) - f'(x)\cdot dx$$

Def: Given the function $f:[a,b]\to\mathbb{R}$ as above, set

$$C = \text{the graph of } f := \{(x, f(x)) \mid a \le x \le b\}$$

For any point $p \in C$ let

$T_p :=$ the straight line tangent to the graph of f through p

Set $T(C) :=$ the tangent bundle of C

$$:= \{(p,q) \mid p \in C \ \& \ q \in T_p\}$$

Then the differentials dx & dy are the functions:

$$dx : T(C) \to \mathbb{R}$$

& $$dy : T(C) \to \mathbb{R}$$

(of 2 variables) defined as follows:

$$dx(p,q) := dx_p(q) := x_q - x_p$$

& $$dy(p,q) := dy_p(q) := y_q - y_p$$

& so, as above, for $q \neq p$,

$$dy(p,q) = f'[x(p)] \cdot dx(p,q)$$

& so

$$dy = f'(x) \cdot dx \qquad \text{(as functions)}.$$

One final comment:

If we use differentials to approximate $(1+x)^n$ for small x we find that

$$(1+x)^n \simeq 1 + nx$$

Proof: Here $f(x) = x^n$

& $f'(x) = nx^{n-1}$

Take $a = 1$ & $x = h$

Then for small h

$$\frac{f(a+h) - f(a)}{h} \simeq f'(a)$$

so

$$f(a+h) - f(a) \simeq f'(a) \cdot h$$

So

$$f(a+h) \simeq f(a) + f'(a) \cdot h$$

i.e.

$$(a+h)^n \simeq a^n + n.a^{n-1}.h$$

& so for $a = 1$ & $x = h =$ small

$$(1+x)^n \simeq 1^n + n \cdot 1^{n-1} \cdot x$$

i.e.,

$$(1+x)^n \simeq 1 + n \cdot x$$

Example: For small x,

$$(1+x)^n \approx 1+nx$$

Hence

$$\sqrt{1+x} = (1+x)^{1/2} \approx 1+\tfrac{1}{2}x = 1+\frac{x}{2}$$

$$\frac{1}{1-x} = (1-x)^{-1} = [1+(-x)]^{-1} \doteq 1+(-1)(-x) = 1+x$$

$$\sqrt[3]{1+5x^4} = [1+5x^4]^{\frac{1}{3}} \approx 1+\tfrac{1}{3}5x^4 = 1+\tfrac{5}{3}x^4$$

$$\frac{1}{\sqrt{1-x^2}} = (1+-x^2)^{-\frac{1}{2}} \approx 1+\left(-\tfrac{1}{2}\right)(-x^2) = 1+\tfrac{1}{2}x^2$$

---

i.e., $\boxed{\sqrt{1+x} \approx 1+\frac{x}{2}}$, $\boxed{\frac{1}{1-x} \approx 1+x}$,

$\boxed{\sqrt[3]{1+5x^4} \approx 1+\frac{5x^4}{3}}$, $\boxed{\frac{1}{\sqrt{1-x^2}} \approx 1+\frac{x^2}{2}}$

Anti derivatives

Problem: Suppose we are given a function

$$f : [a,b] \to \mathbb{R}$$

and asked to find another function

$$F : [a,b] \to \mathbb{R}$$

such that

$$F'(x) = f(x) \quad \text{for each } x \text{ with } a < x < b$$

And suppose such an F exists.
Under these circumstances, F is called an antiderivative of f, or an indefinite integral of f, or simply an integral of f.

We also say that the function

$$y = F(x)$$

is a solution of the differential equation

$$\frac{dy}{dx} = f(x) \quad , \quad a < x < b$$

if over the domain (= open set) $a < x < b$

$F(x)$ is a differentiable function of x

&

$$\frac{d\,F(x)}{dx} = f(x)$$

We then call $F(x)$ an antiderivative of $f(x)$ with respect to x, or an integral of $f(x)$ with respect to x.

Clearly, if $F(x)$ is an integral (= antiderivative) of $f(x)$ w.r.t. (= with respect to) x, so is $F(x) + C$ for any constant C since

$$\frac{d\,[F(x)+C]}{dx} = \frac{d\,F(x)}{dx} + \frac{d\,C}{dx} = f(x) + 0 = f(x)$$

Notation: If $F(x)$ is an integral of $f(x)$ w.r.t. x we symbolize this by writing

$$\int f(x)\,dx = F(x)$$

$$\left[\text{or} \quad \int f(x)\,dx = F(x) + C \quad \right]$$

In short, by definition,

$$\int f(x)\,dx = F(x) \iff \frac{d\,F(x)}{dx} = f(x)$$

The symbol "$\int$" is called an integral sign. Actually, we may think of the symbol

$$\int \ldots dx$$

as meaning "the integral of .... w.r.t. x" just as the symbol $\frac{d \ldots}{dx}$ means "the derivative of .... w.r.t. x"

By this convention, the "$\int$" & the "dx" go together to make up "$\int \ldots dx$" just as the "d" & the "dx" go together to make up the symbol "$\frac{d \ldots}{dx}$"

<u>Remark</u>: We may think of the symbol

"$\int \cdots dx$"

as the <u>inverse</u> of the symbol "$\frac{d \ldots}{dx}$" in the following sense:

By definition

$$\int f(x)\,dx = F(x) \iff \frac{d\,F(x)}{dx} = f(x)$$

Now

$$\frac{d\,F(x)}{dx} = f(x) \implies d\,F(x) = f(x)\,dx$$

so upon applying the integral sign "$\int$" to each side of the "differential equation"

$$d\,F(x) = f(x)\,dx$$

we find that

$$\int d\,F(x) = \int f(x)\,dx = F(x)$$

Upon comparing the left & right sides
of the equation

$$\int dF(x) = \int f(x)\,dx = F(x)$$

we find that

$$\int dF(x) = F(x)$$

Now $dF(x)$ is the <u>differential</u> of $F(x)$
while

$\int dF(x)$ is the <u>integral</u> of the differential of $F(x)$.

The equation

$$\int dF(x) = F(x)$$

tells us that the integral undoes what the differential does
– i.e., that "$\int$" is the "left-inverse" of "d"

Consequences:

1) $\int du = u$

2) $\int c\, du = c \int du$ for any constant c.

3) $\int (du + dv) = \int du + \int dv$

4) $\int u^n\, du = \dfrac{u^{n+1}}{n+1}$, if $n \neq -1$

Proof

1) This is immediate from what we just observed on the last page, namely that the integral undoes what the differential does.

2) $$\frac{d(cu)}{dx} = c\,\frac{du}{dx} \qquad \boxed{*}$$

$$\Downarrow$$

$$d(cu) = c\,\frac{du}{dx}\,.\,dx = c\, du$$

$$\Downarrow$$

$$c\, du = d(cu) \Rightarrow \boxed{d(cu) = c\,du} \; \boxed{**}$$

$$\Downarrow$$

$$\int c\, du = \int d(cu) = cu = c\int du$$

$$\Downarrow$$

$$\int c\, du = c\int du \qquad \boxed{***}$$

Note: $\boxed{*}$ says the derivative of a constant times a function is the constant times the derivative. $\boxed{**}$ & $\boxed{***}$ say the analogous things for "differentials" & for "integrals".

3) $$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad (*)$$

$$\Downarrow$$

$$d(u+v) = \frac{d(u+v)}{dx} \cdot dx$$

$$= \left(\frac{du}{dx} + \frac{dv}{dx}\right) \cdot dx$$

$$= \frac{du}{dx} \cdot dx + \frac{dv}{dx} \cdot dx$$

$$= du + dv$$

Therefore $$d(u+v) = du + dv \qquad (**)$$

so $$\int du + dv = \int d(u+v)$$

$$= u + v$$

$$= \int du + \int dv$$

i.e., $$\int du + dv = \int du + \int dv \qquad (***)$$

Note: (*) says the derivative of a sum is the sum of the derivatives. (**) says the differential of a sum is the sum of the differentials & (***) says the integral of a sum is the sum of the integrals

4) $$\frac{d(u^n)}{dx} = n u^{n-1} \frac{du}{dx}$$

$$\Downarrow$$

$$d(u^n) = n u^{n-1} \frac{du}{dx} . dx = n u^{n-1} du$$

$$\Downarrow$$

$$u^n = \int d(u^n) = \int n u^{n-1} du = n \int u^{n-1} du$$

$$\Downarrow$$

$$\int u^{n-1} du = \frac{u^n}{n} \quad , \text{if } n \neq 0$$

$$\Downarrow$$

$$\int u^m du = \frac{u^{m+1}}{m+1} \quad , \text{if } m \neq -1$$

$$\Downarrow$$

$$\int u^n du = \frac{u^{n+1}}{n+1} \quad , \text{if } n \neq -1 .$$

Examples:

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{x^{3/2}}{3/2}$$

$$= \frac{2}{3}x^{3/2}$$

$$\int \sqrt{2x+1}\,dx = \int (2x+1)^{1/2}\,dx$$

$u = 2x+1$

$\frac{du}{dx} = 2$

$du = 2\,dx$

$$= \int (2x+1)^{1/2} \cdot \frac{2}{2}\,dx$$

$$= \frac{1}{2}\int (2x+1)^{1/2}\,2\,dx$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}$$

$$= \frac{u^{3/2}}{3} = \frac{(2x+1)^{3/2}}{3}$$

Alternatively,

$$\int \sqrt{2x+1}\,dx = \int u^{1/2}\,\frac{1}{2}\,du$$

$u = 2x+1$

$\Downarrow$

$\frac{du}{dx} = 2 \Rightarrow du = 2\,dx$

$\Uparrow$

$dx = \frac{1}{2}du$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} = \frac{u^{3/2}}{3}$$

$$= \frac{(2x+1)^{3/2}}{3}$$

More Formulas

Derivative Formula	Differential Formula	Integral Formula
$\frac{du}{dx} = \frac{du}{dx}$	$du = \frac{du}{dx}dx$	$\int du = \int \frac{du}{dx}dx = u$
$\frac{d(c\cdot u)}{dx} = c\cdot\frac{du}{dx}$	$d(c\cdot u) = c\,du$	$\int c\,du = \int d(c\cdot u)$ $= cu$ $= c\int du$ i.e., $\int c\,du = c\int du$
$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$	$d(u+v) = du + dv$	$\int du + dv = \int d(u+v)$ $= u+v$ $= \int du + \int dv$ i.e., $\int du + dv = \int du + \int dv$
$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}$	$d(u^n) = nu^{n-1}du$	$n\int u^{n-1}du = \int nu^{n-1}du$ $= \int d(u^n)$ $= u^n$ i.e., $\int u^{n-1}du = \frac{u^n}{n}, n \neq 0$ so $\int u^n du = \frac{u^{n+1}}{n+1}, n+1 \neq 0$ $(n \neq -1)$

More Formulas

In the same way, we find that:

$$\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx} \implies d(\sin u) = \cos u \, du$$

Hence

$$\int \cos u \, du = \int d(\sin u) = \sin u$$

i.e.,

$$\boxed{\int \cos u \, du = \sin u}$$

Likewise,

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx} \implies d(\cos u) = -\sin u \, du$$

$$= (-1) \sin u \, du$$

$\therefore$

$$(-1) \int \sin u \, du = \int (-1) \sin u \, du = \int d(\cos u)$$

$$= \cos u$$

Hence

$$\int \sin u \, du = \frac{\cos u}{-1} = (-1) \cos u = -\cos u$$

i.e.

$$\boxed{\int \sin u \, du = -\cos u}$$

By the same reasoning

$$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx} \Longrightarrow d(\tan u) = \sec^2 u\, du$$

$$(t)' = \left(\frac{s}{c}\right)' = \frac{c \cdot s' - s \cdot c'}{c^2}$$
$$= \frac{c \cdot c - s \cdot (-s)}{c^2}$$
$$= \frac{c^2 + s^2}{c^2} = \frac{1}{c^2}$$
$$= \left(\frac{1}{c}\right)^2 = \sec^2$$

so

$$\int \sec^2 u\, du = \int d(\tan u) = \tan u$$

i.e.,

$$\boxed{\int \sec^2 u\, du = \tan u}$$

Likewise,

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx} \Longrightarrow d(\cot u) = -\csc^2 u\, du$$

so $$(-1)\int \csc^2 u\, du = \int -\csc^2 u\, du = \int d(\cot u) = \cot u$$

so $$\int \csc^2 u\, du = \frac{\cot u}{-1} = -\cot u$$

i.e.,

$$\boxed{\int \csc^2 u\, du = -\cot u}$$

Finally,

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx} \Rightarrow d(\sec u) = \sec u \tan u \, du$$

so

$$\int \sec u \tan u \, du = \int d(\sec u) = \sec u$$

i.e.,

$$\boxed{\int \sec u \tan u \, du = \sec u}$$

&

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx} \Rightarrow d(\csc u) = -\csc u \cot u \, du$$

so

$$(-1)\int \csc u \cot u \, du = \int -\csc u \cot u \, du = \int d(\csc u)$$
$$= \csc u$$

so $\displaystyle\int \csc u \cot u \, du = \frac{\csc u}{-1} = -\csc u$

i.e.,

$$\boxed{\int \csc u \cot u \, du = -\csc u}$$