Math 251 Examples on nth derivatives

Factorials: The definition of $n!$ (read "n factorial") is

$$n! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

For example $1! = 1$, $2! = 2\cdot 1 = 2$, $3! = 3\cdot 2\cdot 1 = 6$, $4! = 4\cdot 3\cdot 2\cdot 1 = 24$. We will use the standard convention $\boxed{0! = 1}$ [I really want to emphasize this point, but I'm afraid using exclamation points would just make things more confusing. . .so **REMEMBER:** $0! = 1$.]

Notice $1052! = 1052\cdot 1051!$. Similarly $254! = 254\cdot 253\cdot 252\cdot 251!$. Because of this we see

$$\frac{102!}{99!} = \frac{102\cdot 101\cdot 100\cdot 99!}{99!} = 102\cdot 101\cdot 100,$$

or in general

$$\frac{n!}{r!} = n(n-1)(n-2)\cdots(r+2)(r+1)$$

whenever $r < n$.

Example 1. Let $f(x) = \ln(x)$. Let's look at the first few derivatives:

$$f'(x) = \frac{1}{x}$$

$$f''(x) = \frac{-1}{x^2}$$

$$f'''(x) = \frac{(-2)(-1)}{x^3} = \frac{2}{x^3}$$

$$f''''(x) = \frac{(-3)(-2)(-1)}{x^4} = \frac{-6}{x^4}$$

$$f^{(5)}(x) = \frac{(-4)(-3)(-2)(-1)}{x^5} = \frac{24}{x^5}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(-(n-1))(-(n-2))\cdots(-3)(-2)(-1)}{x^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

So

$$\boxed{f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}}$$

Notice that when $n = 1$ this formula gives

$$f'(x) = \frac{(-1)^0(0)!}{x} = \frac{1}{x}$$

because $0! = 1$.

Example 2. Let $f(x) = xe^x$. We will begin by finding the first derivative using the product rule:

$$f'(x) = \frac{d}{dx}(x)e^x + x\frac{d}{dx}(e^x) = e^x + xe^x = e^x + f(x)$$

So $f'(x) = e^x + f(x)$. Notice we have found a formula for $f'(x)$ which involves $f(x)$. We will use this to discover a pattern in the first few derivatives:

$$f''(x) = \frac{d}{dx}(e^x + f(x)) = \frac{d}{dx}(e^x) + f'(x) = e^x + (e^x + f(x)) = 2e^x + f(x)$$

$$f'''(x) = \frac{d}{dx}(2e^x + f(x)) = \frac{d}{dx}(2e^x) + f'(x) = 2e^x + (e^x + f(x)) = 3e^x + f(x)$$

$$f''''(x) = \frac{d}{dx}(3e^x + f(x)) = \frac{d}{dx}(3e^x) + f'(x) = 3e^x + (e^x + f(x)) = 4e^x + f(x)$$

$$\vdots$$

$$f^{(n)}(x) = \frac{d}{dx}((n-1)e^x + f(x)) = \frac{d}{dx}((n-1)e^x) + f'(x) = (n-1)e^x + (e^x + f(x)) = ne^x + f(x)$$

Since $f(x) = xe^x$ we see $\boxed{f^{(n)}(x) = ne^x + xe^x}$

Example 3. Let $f(x) = \sin(x)$. In class we noticed that

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

$$f''''(x) = \sin(x)$$

So that $f''''(x) = f(x)$, thus if we listed higher derivatives, we would be repeating the four expressions $\cos(x)$, $-\sin(x)$, $-\cos(x)$, $\sin(x)$ over and over again.

Now remember from trigonometry $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$, thus

$$\sin\left(\frac{\pi}{2} + x\right) = \cos(x)$$

$$\sin\left(\frac{2\pi}{2} + x\right) = -\sin(x)$$

$$\sin\left(\frac{3\pi}{2} + x\right) = -\cos(x)$$

$$\sin\left(\frac{4\pi}{2} + x\right) = \sin(x)$$

$$\vdots$$

Thus we can write the nth derivative as follows:

$$f^{(n)}(x) = \sin\left(\frac{n\pi}{2} + x\right).$$

Example 4. Let $f(x) = x\sin(x)$. If you find the first few derivatives, you should get:

$$f'(x) = x\cos(x) + \sin(x)$$

$$\begin{aligned}
f''(x) &= -x\sin(x) + 2\cos(x) \\
f'''(x) &= -x\cos(x) - 3\sin(x) \\
f''''(x) &= x\sin(x) - 4\cos(x) \\
f^{(5)}(x) &= x\cos(x) + 5\sin(x) \\
f^{(6)}(x) &= -x\sin(x) + 6\cos(x) \\
f^{(7)}(x) &= -x\cos(x) - 7\sin(x) \\
f^{(8)}(x) &= x\sin(x) - 8\cos(x) \\
f^{(9)}(x) &= x\cos(x) + 9\sin(x) \\
&\vdots
\end{aligned}$$

You should convince yourself (using addition identities for sine and cosine) that the following formula gives the nth derivative:

$$f^{(n)}(x) = x\sin\left(\frac{n\pi}{2} + x\right) - n\cos\left(\frac{n\pi}{2} + x\right)$$

Exercises: Here are a few exercises on nth derivatives which might be fun for you to do. I won't be collecting them for credit, but I will be happy to look over your solutions.

1. Find a formula for the nth derivative of the following functions.

   (a) $f(x) = \cos(x)$ (Hint: look at example 3)

   (b) $f(x) = x\cos(x)$ (Hint: look at example 4)

   (c) $f(x) = x\ln(x)$ for $n > 1$ (Hint: use example 1)

   (d) $f(x) = e^{5x}$

   (e) $f(x) = x^n$

   (f) $f(x) = x^k$ where k is some integer greater than n.

2. Let

   $$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

   be an arbitrary polynomial of degree n. [i.e. the a_i's are constants]

   (a) Find a formula for $f^{(m)}(x)$ where $m > n$.

   (b) Find a formula for $f^{(m)}(x)$ where $m = n$.

   (c) Find a formula for $f^{(m)}(x)$ where $m < n$. (this one is harder)