2221-2 Test 1
Tuesday, October 7th, 2008

Name

Student ID #

Instructions. The test runs for an hour - 9:30am to 10:30am - it is closed book, but calculators are allowed. Show all your work on the problem page, or indicate clearly if you write on another page. Partial credit is given for working, full credit is given for correct answers with justification, but no credit is given without working.

There are 4 problems, each worth 5 points. If you have any questions, come and ask.

This box is for the grader's use only - do not write answers here.

1. .... $2\frac{1}{2}$ ....
2. .... 5 ....
3. .... 4 ....
4. .... 5 ....

T. .... $16\frac{1}{2}$ ....

$S + cc = 1$

$\cos^2 x = 1 - \sin^2 x$

$\frac{dy}{dx} + P(x)\, y = Q x$ $\qquad$ $\frac{1}{\cos x}\left(\cos x \frac{dy}{dx} - y \sin x = \frac{\sin x}{\cos x}\right)$

1. Find the solution to the initial value problem

$$\cos x \frac{dy}{dx} - y \sin x = \tan x$$

$$y(\pi/4) = 0.$$

$\frac{dy}{dx} - y \tan x = \frac{\sin x}{1 - \sin^2 x}$

$\frac{dy}{dx} = \frac{\sin x}{1 - \sin^2 x} -$

S

$\frac{\partial y}{\partial x} - \frac{y \sin x}{\cos x} = \frac{\sin x}{\cos x \cos^2}$

$\frac{dy}{dx} - y \tan x = \frac{\sin x}{1 - \sin^2 x}$

$P(x) = -\tan x$

$\mu(x) = e^{\int -\tan x\, dx} = -e^{\cancel{\cos^2 x}}$

$\frac{\partial y}{\partial x} - y \tan x = \frac{\sin x}{\cos^2 x}$

$\cos x\, dy = (\tan x - y \sin x)\, dx$ (N, M)

$\frac{\partial N}{\partial x} = -\sin x \qquad \frac{\partial M}{\partial y} = -\sin x$

Exact

$\mu(x) = e^{-\int \tan x\, dx} \Rightarrow \mu x = e^{-\sec^2 x}$ X

$\frac{\partial y}{\partial x} = \frac{\tan x - y \sin x}{\cos x}$

$-\int \frac{\sin x}{\cos x}$ $\qquad$ $p = \cos x$ $\qquad$ $\frac{\partial p}{\partial x} = -\sin x\, dx$

$\frac{\partial y}{\partial x} = \frac{\tan x}{\cos x} - \frac{y \sin x}{\cos x}$

$\int \frac{1}{p}\, dp$

$\ln p$

$\ln|\cos x|$

$\frac{dy}{dx} + P(x) y = Q x$

$\frac{\partial y}{\partial x} - y \tan x = \frac{\sin x}{\cos^2 x}$

$\frac{\partial y}{\partial x} \cos x - y \sin x = \frac{\sin x}{\cos^2 x} (\cancel{\cos x})$

$\mu x = e^{\ln|\cos x|} = \cos x = \mu x$

$y = \frac{\ln|\cos x| + \frac{\sqrt{2}}{2}}{\cos x}$

$\int \cancel{d} y \cos x = \int \tan x\, dx$ X

$y \cos x = \ln|\cos x| + C$

$y = \frac{\ln|\cos x| + C}{\cos x}$ X $\Rightarrow y = \frac{\ln \cos(\frac{\pi}{4}) + C}{\cos \frac{\pi}{4}} \Rightarrow 0 = \ln\left|\frac{\sqrt{2}}{2}\right| + C$

$C = \ln \frac{\sqrt{2}}{2}$

$2\frac{1}{2}$

2. Solve the differential equation

$$(\sin x + xy^2)dx + (\cos y + x^2y)dy = 0, \;\; y(\sqrt{\pi}) = 1$$

M $\quad$ N

$$\frac{\partial M}{\partial y} = 2yx \qquad \frac{\partial N}{\partial \cancel{x}} = 2xy$$

~~not exact~~

exact

$\int \cos y + x^2 y \, dy$

$-\sin y + x^2 \frac{1}{2} y^2$

$-\sin y + \frac{1}{2}$

$\int M \, dx = \int \sin x + xy^2 \, dx$

$\cos x + \frac{1}{2}x^2y^2 + g(y)$

$\frac{\partial}{\partial y}\left(\cos x + \frac{1}{2}x^2y^2 + g(y)\right)$

$\frac{1}{2}x^2 2y + g'y$

$\cancel{x^2y} + g'y = \cos y + \cancel{x^2y}$

$\int g'y = \int \cos y$

$g(y) = -\sin y$

$\cos x + \frac{1}{2}x^2y^2 - \sin y = y(x)$

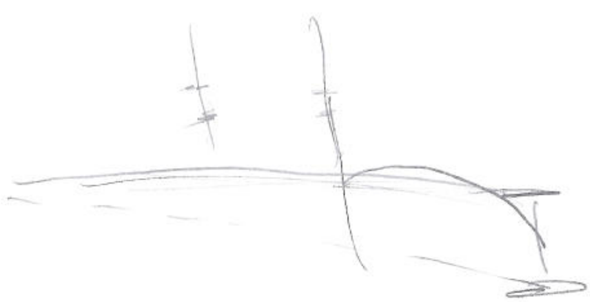
$1 = \cos\sqrt{\pi} + \frac{1}{2}\sqrt{\pi}^2(1)^2 - \sin 1 = 1$

$\boxed{1 = \cos\sqrt{\pi} + \frac{\pi}{2} - \sin(1)}$

$\cos x + \frac{1}{2}x^2y^2 - \sin y = C$

$\cos\sqrt{\pi} + \frac{1}{2}\sqrt{\pi}^2(1) - \sin(1) = C$

$\boxed{\cos\sqrt{\pi} + \frac{\pi}{2} - \sin(1) - 1 = y(t)}$



$\cos\sqrt{\pi} + \frac{\pi}{2} - \sin(1) = C$

5

3. Solve the Bernoulli equation

$$\frac{dy}{dx} + \frac{y}{x} - x^3y^2 = 0$$

$$\frac{\partial y}{\partial x} + P(x)y = Q(x)y^n$$

$$n = 2$$

$$\frac{\partial y}{\partial x} + \frac{y}{x} = x^3y^2$$

$$v = y^{1-n}$$

$$-y^2\frac{\partial v}{\partial x} + \left(\frac{y}{x}\right) = x^3y^2$$

$$v = y^{1-2}$$

$$\frac{1}{-y^2}\left[-y^2 \frac{\partial v}{\partial x} + \left(\frac{y}{x}\right) = \left(x^3y^2\right)\right]$$

$$v = y^{-1}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{y^2}\frac{\partial y}{\partial x}$$

$$\frac{\partial v}{\partial x} - \frac{1}{xy} = x^3$$

$$-y^2\frac{\partial v}{\partial x} = \frac{\partial y}{\partial x}$$

$\frac{\partial v}{\partial x} + P(x)v = Q(x)$ linear

$$\frac{\partial v}{\partial x} - \frac{v}{x} = -x^3$$

$$P(x) = -\frac{1}{x}$$

$$\frac{\partial v}{\partial x}x^{-1} = x^3(x^{-1})$$

$$\mu(x) = e^{-\int\frac{1}{x}\partial x} = e^{-\ln|x|}$$

$$\int \partial v\, x^{-1} = \int x^2\,\partial x$$

$$\mu(x) = x^{-1}$$

$$v = \frac{1}{3}x^3(x^1)$$

$$v = -\frac{1}{3}x^4 + \text{const}\cdot x$$

$$y^{-1} = \frac{1}{3}x^4$$

$$\frac{1}{\frac{1}{3}x^4} = \boxed{y = 3x^{-4}}$$

4

$c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$

$c_1 e^{\alpha t}\beta\cos\beta t + \alpha c_1 e^{\alpha t}\cos\beta + \alpha c_2 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\beta\sin\beta t$

$\alpha = -1 \quad c_2 = 0$
$\beta = 1 \quad c_1 = -e^{\frac{\pi}{2}}$

4. Solve the problem

$$y'' + 2y' + 2y = 0$$
$$y(\pi/2) = 0, y'(\pi/2) = 1$$

$-e^{\frac{\pi}{2}} e^{(-t)}\cos t = y(t)$

$\boxed{-e^{(\frac{\pi}{2}-t)}\cos t = y(t)}$

$r^2 + 2r + 2$

$c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t = y(t)$

$c_1 e^{(-1)(\frac{\pi}{2})}\cos(1)\frac{\pi}{2} + c_2 e^{1\frac{\pi}{2}}\sin(1)(\frac{\pi}{2}) = 0$

$a=1$
$b=2$
$c=2$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\frac{-2 \pm \sqrt{4-(4)(1)(2)}}{2(1)}$

$\frac{-2 \pm \sqrt{-4}}{2}$

$-2 \pm 2\sqrt{2}\,i$

$c_1 e^{-\frac{\pi}{2}}\cos\frac{\pi}{2} + c_2 e^{\frac{\pi}{2}}\sin\frac{\pi}{2} = 0$

$0 + c_2 e^{\frac{\pi}{2}} = 1$

$c_2 = \frac{1}{e^{\pi/2}} = e^{-\frac{\pi}{2}} = c_2$

$c_1 e^{-\frac{\pi}{2}}\cos\frac{\pi}{2} + e^{-\frac{\pi}{2}}$

$-c_1 e^{(\alpha t)}\sin(\beta t)(\beta) + \alpha c_1 e^{\alpha t}\cos\beta t + (\alpha)c_2 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\cos\beta t(\beta)$

$1 = \beta = \frac{\sqrt{4ac-b^2}}{2a} = \frac{\sqrt{4(1)(2)-4}}{2(1)}$

$\frac{\sqrt{4}}{2} = 1$

$-c_1 e^{-t}\sin\frac{\pi}{2}(1) + (-1)(c_1)e^{(-\frac{\pi}{2})}\cos$

$-1 = \alpha = \frac{-b}{2a} \quad \frac{-2}{2(1)} = -1$

$-c_1 e^{\alpha t}\sin(\beta t)\beta + \alpha c_1 e^{\alpha t}\cos\beta t + \alpha c_2 e^{\alpha t}\sin\beta t + c_2 e^{\alpha t}\cos\beta t\,\beta$

$-c_1 e^{-\frac{\pi}{2}}\underbrace{\sin(\tfrac{\pi}{2})}_{1} + (-1)c_1 e^{-\frac{\pi}{2}}\cos(\tfrac{\pi}{2}) + (-1)c_2 e^{-\frac{\pi}{2}}\sin\frac{\pi}{2} + c_2 e^{-\frac{\pi}{2}}\cos\frac{\pi}{2}(1) = 1$

$\cancel{c_1 e^{-\frac{\pi}{2}} + 0 + (-1)e^{-\frac{\pi}{2}}e^{\frac{\pi}{2}}(1) + 0} = 1$

$(-e^{\frac{\pi}{2}} - c_2)e^{-\frac{\pi}{2}}\cos\frac{\pi}{2} + c_2 e^{\frac{\pi}{2}}\sin\frac{\pi}{2} = 0$

$-c_1 e^{-\frac{\pi}{2}}(1) + 0 - c_2 e^{-\frac{\pi}{2}}(1) + 0 = 1$

$(-1 - c_2 e^{-\frac{\pi}{2}})0 + c_2 e^{\frac{\pi}{2}}\sin\frac{\pi}{2}$

$-c_1 e^{-\frac{\pi}{2}} - c_2 e^{-\frac{\pi}{2}} = 1$

$c_2 e^{\frac{\pi}{2}} = 0$

$-e^{-\frac{\pi}{2}}(c_1 + c_2) = 1$

$c_2 = 0$

5

$c_1 + c_2 = -e^{\frac{\pi}{2}}$

$c_1 = -e^{\frac{\pi}{2}} - c_2$