2221-2 Test 2
Tuesday, November 18th, 2008

Name

Student ID #

Instructions. The test runs for one hour - 9:30am to 10:30am - it is closed book, but calculators and a table of transforms are allowed. Show all your work on the problem page, or indicate clearly if you write on another page. Partial credit is given for working, full credit is given for correct answers with justification, but no credit is given without working.

There are 4 problems, each worth 5 points. If you have any questions, come and ask.

This box is for the grader's use only - do not write answers here.

1. .... 5 ....
2. .... 5 ....
3. .... 4½ ....
4. .... ½ ....

T. .... 15 ....

1. Use the *method of undetermined coefficients* to solve the initial value problem

$$y'' + 2y' + y = e^{-t}$$
$$y(0) = 1/2, \ \ y'(0) = 0.$$

$ct^n e^{rt}$ $\quad s=2$, $n=0$, $r=-1$

r^2+2r+1

$(r+1)(r+1)$

$r=-1$

double

$s=2$

$t^2(A_m t^n + A_0)e^{rt}$

$t^2 A e^{-t}$

$y\,t = \cancel{y_p t + y_h t}$

$y(t) = y_p t + y_h t$

1 $\quad y_p t = At^2e^{-t}$

2 $\quad y_p' t = 2At e^{-t} - 2At^2 e^{-t}$ $\quad 4$

1 $\quad y_p'' t = 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t}$

$-2Ate^{-t} + At^2e^{-t}$

$2Ae^{-t} - \cancel{2Ate^{-t}} - \cancel{2Ate^{-t}} + \cancel{At^2e^{-t}} + \cancel{4Ate^{-t}} - \cancel{2At^2e^{-t}} + \cancel{At^2e^{-t}}$

$2A\cancel{e^{-t}} = \cancel{e^{-t}}$

$2A = 1$

$A = \frac{1}{2}$

$\underline{y_p t = \frac{1}{2}t^2e^{-t}}$

$y\,t = \frac{1}{2}t^2e^{-t} + c_1e^{-t} + c_2te^{-t}$

$y(0) = \frac{1}{2} = \frac{1}{2}(0)e^0 + c_1 + 0$

$\underline{c_1 = \frac{1}{2}}$

$y'(t) = \frac{1}{2}2te^{-t} - \frac{1}{2}t^2e^{-t} - c_1e^{-t} + c_2e^{-t} - c_2te^{-t} = 0$

$0 \quad 0 \quad -c_1 + c_2 - 0 = 0$

$-\frac{1}{2} + c_2 = 0 \Rightarrow \underline{c_2 = \frac{1}{2}}$

$$\boxed{y(t) = \frac{1}{2}t^2e^{-t} + \frac{1}{2}e^{-t} + \frac{1}{2}te^{-t}}$$

$y(t) = \frac{1}{2}e^{-t}(t^2+t+1)$

5

2. Use *variation of parameters* to solve the initial value problem

$$y'' + 2y' + y = t^{-1}e^{-t}$$

$$y(1) = 1, \; y'(1) = 0.$$

$y_1 v_1' + y_2 v_2' = 0$

$y_1' v_1' + y_2' v_2' = \cancel{g}/A$

$y_p t = y_1 v_1 + y_2 v_2$

$r^2 + 2r + 1$

$(r+1)(r+1)$

$r = -1$

$s = 2$

$c_1 e^{rt} + c_2 t e^{rt}$

$e^{-t} + te^{-t}$

$y_1 \quad y_2$

$y_1 = e^{-t} \qquad y_1' = -e^{-t}$

$y_2 = te^{-t} \qquad y_2' = e^{-t} - te^{-t}$

$y_1 v_1' + y_2 v_2' = 0$

$e^{-t} v_1' + te^{-t} v_2' = 0$

$\cancel{e^{-t}} v_1' = -t\cancel{e^{-t}} v_2'$

$v_1' = t v_2'$

$-e^{-t} v_1' + (e^{-t} - te^{-t}) v_2' = t^{-1} e^{-t}$

$\cancel{e^{-t}}(t v_2') + v_2' \cancel{e^{-t}} - t\cancel{e^{-t}} v_2' = t^{-1} \cancel{e^{-t}}$

$t v_2' + v_2' - t v_2' = t^{-1}$

$v_2'(t + 1 - t) \quad \underline{v_2' = t^{-1}}$

$\int v_2' = \int t^{-1}\,dt$

$v_2 = \ln t$ ✓

$\underline{v_1' = t(t^{-1})}$

$\int v_1' = \int 1\,dt$

$v_1 = -t$

$\underline{y_p t = -te^{-t} + te^{-t} \ln t}$

$y(1)$

$y(1) = te^{-t} + te^{-t}\ln t + c_1 e^{-t} + c_2 te^{-t} = 1$

$e^{-1} + e^{-1}(0) + c_1 e^{-1} + c_2 e^{-1} = 1$

$e^{-1}(1 + c_1 + c_2) = \underline{\frac{1}{e^{-1}} - 1 - c_2 = c_1}$

$y(t) = te^{-t} + te^{-t}\ln + c_1 e^{rt} + c_2 te^{rt}$

$\rightarrow \frac{1}{e^{-1}} - 1 + \frac{(2e^{-1} - 1)}{e^{-1}} = c_1$

$c_1 = 2e^{-1} - 1$

$y'(1) = e^{-t} - te^{-t} + e^{-t}\ln t - te^{-t}\ln t + te^{-t}\frac{1}{t} - e^{-t}c_1 + c_2 e^{-t} - c_2 te^{-t} = 0$

$\cancel{e^{-1}} - \cancel{e^{-1}} + \cancel{e^{-1}(0)} - \cancel{e^{-1}(0)} + e^{-1} - e^{-1}c_1 + e^{-1}c_2 - c_2 e^{-1} = 0$

$e^{-1} - e^{-1}\left(\frac{1}{e^{-1}} - 1 - c_2\right) + \cancel{e^{-1}c_2} - \cancel{c_2 e^{-1}} = 0$

$e^{-1} - 1 + e^{-1} + c_2 e^{-1} + \cancel{e^{-1}c_2} - \cancel{c_2 e^{-1}} = 0$

$y(t) = te^{-t} + te^{-t}\ln t + c_1 e^{rt} + c_2 te^{rt}$

$c_2 = \frac{-2e^{-1} + 1}{e^{-1}}$

$\boxed{y(t) = te^{-t} + te^{-t}\ln t + (2e^{-1} - 1)e^{-t} + \left(\frac{-2e^{-1}+1}{e^{-1}}\right)te^{-t}}$

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More work on back of prev pg.

3. *Prove* the result that

$$\mathcal{L}\{e^{at}\cos bt\}(s) = \frac{s-a}{(s-a)^2+b^2},\ s > a.$$

(You may use result #2 from the tables: $\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a)$, where $F(s)$ is the Laplace transform of $f(t)$.)

$u \quad du \qquad v \quad dv$

$$\mathcal{L} = \int_0^\infty e^{-st} f_t \, dt$$

$$\mathcal{L} = \int_0^n e^{-st} e^{at} \cos bt \, dt = \int_0^n e^{-t(s-a)} \cos bt \, dt$$

$u \ \cos bt \qquad v \ -\frac{1}{(s-a)} e^{-t(s-a)}$

$du - b\sin bt \qquad dv \ e^{-t(s-a)}$

$$I = \cos bt \left(-\frac{1}{s-a} e^{-t(s-a)}\right) - \int -\left(\frac{1}{s-a}\right) e^{-t(s-a)} - b \sin bt \, dt$$

$$I = -\frac{1}{s-a} \cos bt \, e^{-t(s-a)} - \left(\frac{b}{s-a}\right) \int e^{-t(s-a)} \sin bt \, dt$$

$p \ \sin bt \qquad k \ -\frac{1}{(s-a)} e^{-t(s-a)}$

$dp \ b\cos bt \qquad dk \ e^{-t(s-a)}$

$$\left[\sin bt \left(-\frac{1}{(s-a)}\right) e^{-t(s-a)} - \int b \cos bt \left(-\frac{1}{(s-a)}\right) e^{-t(s-a)} dt\right.$$

I

$$I = -\frac{1}{s-a} \cos bt \, e^{-t(s-a)} - \frac{b}{s-a} \left[-\frac{1}{(s-a)} \sin bt \, e^{-t(s-a)} + \frac{b}{(s-a)} \int \cos bt \, e^{-t(s-a)}\right.$$

$$= -\frac{1}{(s-a)} \cos bt \, e^{-t(s-a)} + \frac{b}{(s-a)^2} \sin bt \, e^{-t(s-a)} \cancel{- \frac{b^2}{(s-a)^2} I}$$

explain $s > a$

$$\frac{(s-a)^2}{(s-a)^2} I + I \left(\frac{b^2}{(s-a)^2}\right) = \frac{1}{s-a} \left(-\cos bt \, e^{-t(s-a)} \Big|_0^\infty + \frac{b}{(s-a)} \sin b \, e^{-t(s-a)} \Big|_0^\infty\right.$$

$$\cancel{\left(\frac{(s-a)^2 + b^2}{(s-a)^2}\right)} I = \frac{1}{\cancel{s-a}} \frac{(s-a)^{\cancel{2}}}{(s-a)^2 + b^2} \left[0 - (-1) + 0 - 0\right]$$

$$\boxed{I = \frac{(s-a)}{(s-a)^2 + b^2}}$$

$4\frac{1}{2}$

4. Use the *Laplace transform* to solve the initial value problem

$$y'' - y = t^2$$

$$y(0) = 0, \;\; y'(0) = 0.$$

y'' $s^2 Y(s) - y(0)s - y'(0)$

$~~y'~~$ $sY(s) - y(0)$

$s^2 Y(s) - 0(s) - 0 - [Y(s)s]$ $\quad Y \quad Y(s)$

$s^2 Y(s) - Y(s) = \cancel{t}^2$ X

$Ys(s^2-1) = \frac{t^2}{(s^2-1)}$ $\qquad \frac{t^2}{(s^2-1)}$

$Y = \frac{\cancel{t^2}}{\cancel{s^2-s}}$ $\quad \frac{A}{(s+1)} + \frac{B}{(s-1)}$

(s^2-1)

$(s+1)(s-1)$

$B(s+1) \quad Bs+B$

$A(s-1) \quad As-A$

$(B+A)s = 0$

A

$\frac{t^2}{}$

$\frac{\cancel{t}^2}{(s^2-1)}$

$Ys^2 - y(0)s - y'(0)$

$Ys - y_0$

Y

$Ys^2 - 0 - 0 - Y$

$Y(s^2-1) = t^2$

$Y = \frac{t^2}{(s^2-1)}$ $\quad \frac{A}{s+1} + \frac{B}{(s-1)}$ $\quad B = A+1$

$(s+1)(s-1)$

$B(s+1) \quad Bs+B$ $\quad (B+A) = 1$

$A(s-1) \quad As-A$ $\quad s(B+A) = 0$

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$s(A+1+A) \quad s(2A+1) = 0$