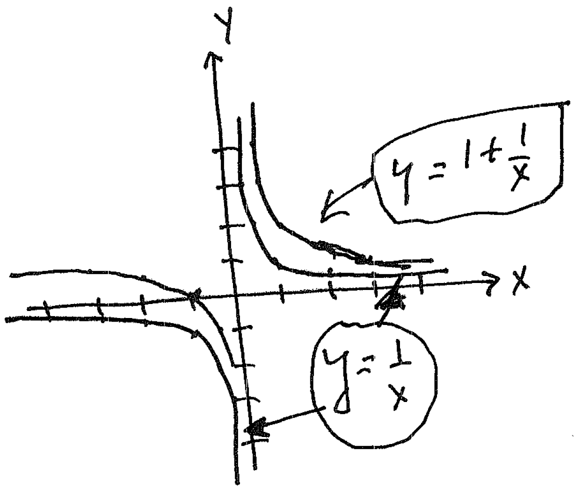
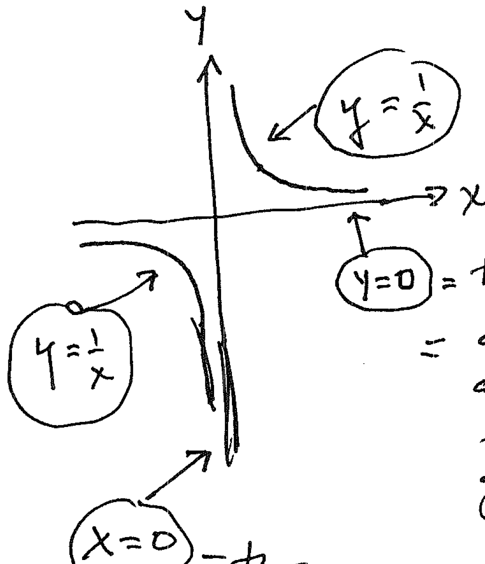
Math 2107

I. (15). $\lim\limits_{x \to 0^-} \left(1 + \frac{1}{x}\right) := \lim\limits_{\substack{x \to 0 \\ x < 0}} \left(1 + \frac{1}{x}\right) = ?$

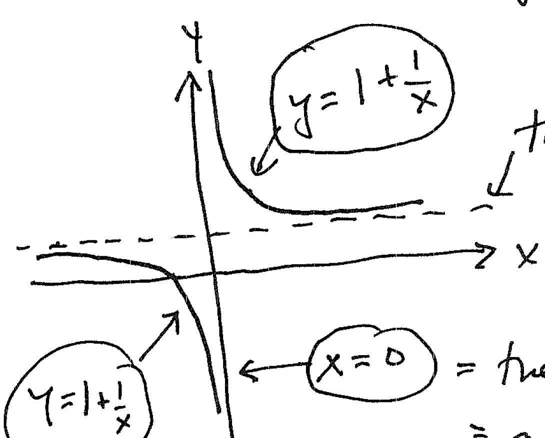


x	$\frac{1}{x}$	$1 + \frac{1}{x}$
1	1	2
2	$\frac{1}{2}$	$1 + \frac{1}{2} = 3/2$
3	$\frac{1}{3}$	$1 + \frac{1}{3} = 4/3$
$\frac{1}{2}$	2	$1 + 2 = 3$
$\frac{1}{3}$	3	$1 + 3 = 4$
-1	-1	$1 + -1 = 0$
-2	$\frac{1}{-2}$	$1 - \frac{1}{2} = \frac{1}{2}$
-3	$\frac{1}{-3}$	$1 - \frac{1}{3} = \frac{2}{3}$
$-\frac{1}{2}$	-2	$1 - 2 = -1$
$-\frac{1}{3}$	-3	$1 - 3 = -2$



$(y = 0)$ = the x-axis = a horizontal asymptote to the graph of $y = \frac{1}{x}$

$(x = 0)$ = the y-axis = a vertical asymptote to the graph of $y = \frac{1}{x}$



the line $(y = 1)$ = a horizontal asymptote to the graph of $y = 1 + \frac{1}{x}$

$(x = 0)$ = the y-axis = a vertical asymptote to the graph of $y = 1 + \frac{1}{x}$

<u>Notice</u> that the graph of

$$y = 1 + \frac{1}{x} = \frac{1}{x} + 1$$

may be obtained from the graph of

$$y = \frac{1}{x}$$

by shoving the graph of $y = \frac{1}{x}$ up, vertically, 1 unit !!!

so both graphs have the y-axis (whose equation is $x = 1$) as a vertical asymptote.

As you can see

$$\lim_{x \to 0^-} \left(1 + \frac{1}{x}\right) = -\infty = \lim_{x \to 0^-} \frac{1}{x}$$

while

$$\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right) = +\infty = \lim_{x \to 0^+} \frac{1}{x}$$

So $\lim\limits_{x \to 0} \left(1 + \frac{1}{x}\right)$ does not exist.

I. 1b

$$\underset{x\to 0}{L}\ \frac{\sin(4x)}{5x} = \underset{x\to 0}{L}\ \sin(4x)\cdot\frac{1}{5x}$$

$$= \underset{x\to 0}{L}\ \frac{4x}{4x}\cdot \sin(4x)\cdot\frac{1}{5x}$$

$$= \underset{x\to 0}{L}\ 4x\cdot\frac{\sin(4x)}{4x}\cdot\frac{1}{5x}$$

$$= \underset{x\to 0}{L}\ \frac{4x}{5x}\cdot\frac{\sin(4x)}{4x}$$

$$= \underset{x\to 0}{L}\ \frac{4}{5}\cdot\frac{\sin(4x)}{4x}$$

$$= \underset{x\to 0}{L}\ \frac{4}{5}\cdot \underset{x\to 0}{L}\ \frac{\sin(4x)}{4x}$$

$$= \underset{x\to 0}{L}\ \frac{4}{5}\ \underset{4x\to 0}{L}\ \frac{\sin(4x)}{4x}$$

$$= \frac{4}{5}\cdot 1 = \frac{4}{5}$$

Also,

$$\underset{x\to 0}{L}\ \frac{\sin(4x)}{5x} = \underset{x\to 0}{L}\ \frac{4}{5}\ \frac{\sin(4x)}{4x}$$

$$= \frac{4}{5}\cdot \underset{x\to 0}{L}\ \frac{\sin(4x)}{4x} = \frac{4}{5}\cdot 1 = 0$$

Note: $\sin(4x) \neq 4\sin x$ since $|\sin x| \leq 1,\ \forall x \in \mathbb{R}$

if $\sin 4x = 4\sin x$, then in particular for $x = \frac{\pi}{2}$

$4x = 4\cdot\frac{\pi}{2} = 2\pi$ & $0 = \sin(2\pi) = \sin\left(4\cdot\frac{\pi}{2}\right) = 4\cdot\sin\left(\frac{\pi}{2}\right)$
$= 4\cdot 1 = 4$

so $0 = 4$!!!

I. 17

$$\lim_{x\to+\infty} \frac{x}{\sqrt{4x^2-1}} = \lim_{x\to+\infty} \frac{x}{\sqrt{4x^2\cdot\left(1-\frac{1}{4x^2}\right)}}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{4x^2}\cdot\sqrt{1-\frac{1}{4x^2}}}$$

$$= \lim_{x\to+\infty} \frac{x}{\sqrt{(2x)^2}\cdot\sqrt{1-\frac{1}{4x^2}}}$$

$$= \lim_{x\to+\infty} \frac{x}{|2x|\cdot\sqrt{1-\frac{1}{4x^2}}}$$

$$= \lim_{x\to+\infty} \frac{x}{|2|\cdot|x|\cdot\sqrt{1-\frac{1}{4x^2}}}$$

$$= \lim_{x\to+\infty} \frac{x}{2\cdot|x|\cdot\sqrt{1-\frac{1}{4x^2}}}$$

$$= \lim_{x\to+\infty} \frac{x}{2\cdot x\cdot\sqrt{1-\frac{1}{4x^2}}}$$

because

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

$$= \lim_{x\to+\infty} \frac{1}{2\cdot\sqrt{1-\frac{1}{4x^2}}}$$

$$= \frac{1}{2\cdot\sqrt{1-0}} = \frac{1}{2}$$

I 18 $$\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{\tan x} = \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} \cdot \frac{\frac{\cos x}{1}}{\frac{\cos x}{1}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\cos x} \cdot \frac{\cos x}{1}}{\frac{\sin x}{\cos x} \cdot \frac{\cos x}{1}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\sin x}$$

$$= \frac{1}{\lim_{x \to \frac{\pi}{2}} \sin x} = \frac{1}{1} = 1$$

19 $$\lim_{x \to 0} \tan x \cdot \cot x = \lim_{x \to 0} \frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x}$$

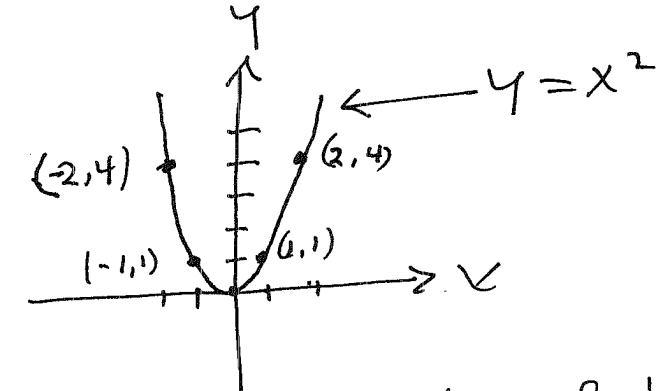
$$= \lim_{x \to 0} 1 = 1$$

I. 20

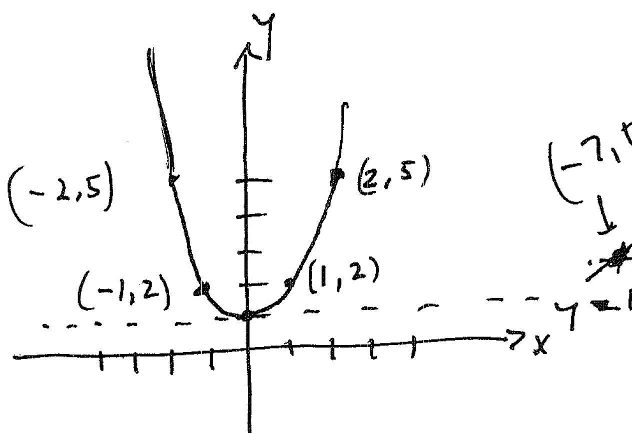
$$f(x) = \begin{cases} x^2+1, & \text{if } x < -2 \\ x+7, & \text{if } x \geq -2 \end{cases}$$

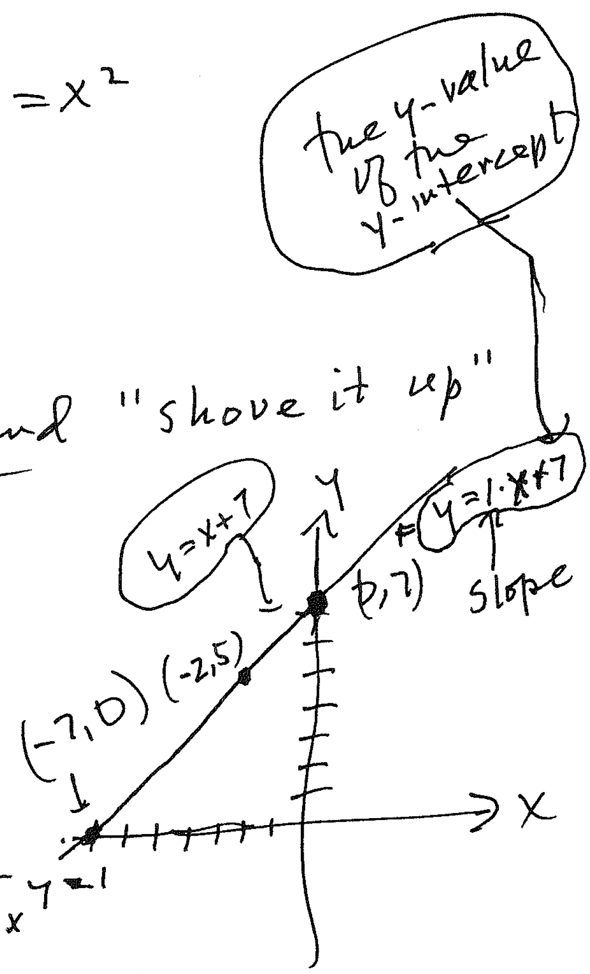
$$\Downarrow$$

$$\lim_{x \to -2} f(x) = ?$$

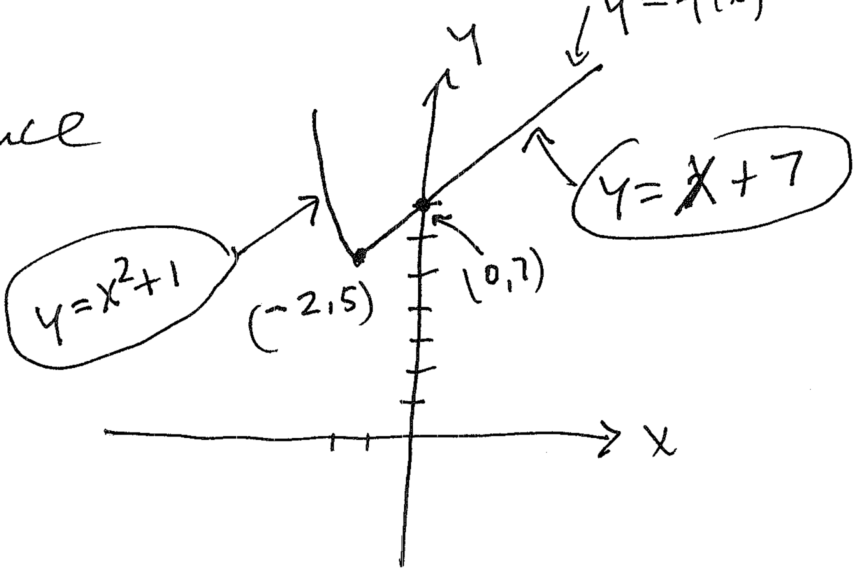


Take this graph and "shove it up" one unit to get





Hence



Therefore $\lim_{x \to -2} f(x) = 5$.

Alternatively,

$$\lim_{x \to (-2)^-} f(x) = \lim_{\substack{x \to -2 \\ x < -2}} f(x) = \lim_{\substack{x \to -2 \\ x < -2}} (x^2+1)$$

$$= \lim_{\substack{x \to -2 \\ x < -2}} x^2 + \lim_{\substack{x \to -2 \\ x < -2}} 1$$

$$= (-2)^2 + 1$$

$$= 4 + 1 = 5$$

while

$$\lim_{x \to (-2)^+} f(x) = \lim_{\substack{x \to -2 \\ x > -2}} f(x)$$

$$= \lim_{\substack{x \to -2 \\ x > -2}} x + 7$$

$$= \lim_{\substack{x \to -2 \\ x > -2}} x + \lim_{\substack{x \to -2 \\ x > -2}} 7$$

$$= -2 + 7 = 7 - 2 = 5,$$

Since $\lim_{x \to (-2)^-} f(x) = 5 = \lim_{x \to (-2)^+} f(x)$

it follows that $\lim_{x \to -2} f(x)$ exists and moreover

$$\lim_{x \to -2} f(x) = \lim_{x \to (-2)^-} f(x) = \lim_{x \to (-2)^+} f(x) = 5$$