Math 2107

Solutions of the Max-Min. Word Problems

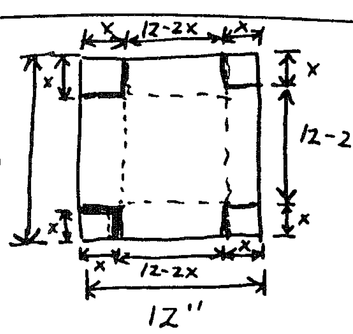
Procedure: Step 1: Draw a picture
Label the parts that are important

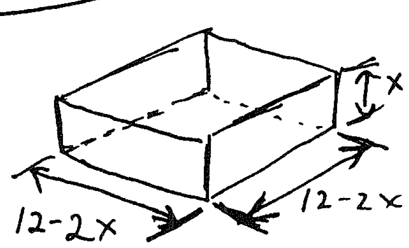
Step 2: Write an equation for the quantity to be maximized or minimized. If possible express that quantity as a function of a single variable. That may require some algebra & the use of information provided in the statement of the problem. Sometimes implicit differentiation is a wise choice. Note the domain of the function.

Step 3: Test the critical points & the end points for candidates of possible max. or min. points. Then use the sign of f' &/or f'' to determine which values of f represent max. or min.

Problem 1: Step 1

V = Volume = L.W.H





V = Length · Width · Height

$V(x) = (12-2x)\cdot(12-2x)\cdot x$

Step 2 $\therefore V(x) = 2(6-x)\cdot 2\cdot(6-x)\cdot x = (-2)(x-6)(-2)(x-6)\cdot x$

$= (-2)^2\cdot(x-6)^2\cdot x = 4x(x^2-12x+36) = 4x^3-48x^2+144x$

$$V(x) = (12-2x)\cdot(12-2x)\cdot x$$

$$= (-2)(x-6)(-2)(x-6)\cdot x$$

$$= 4x\cdot(x-6)^2 = 4x\cdot(x^2-12x+36)$$

$$= 4x^3 - 48x^2 + 144x \quad , \quad 0 \le x \le 6 \text{. (Why?)}$$

You can't cut away a negative amount of material or more than is present!

$V(0) = 0$ & $V(6) = 0$.

$V'(x) = 12x^2 - 96x + 144 \quad \forall x$

$\therefore$ (Since $V'(x)$ exists $\forall x$), $V'(x) = 0$ when $V(x)$ has a max. or a min.

Now $\quad V'(x) = 12\cdot(x^2 - 8x + 12) = 12\cdot(x-2)\cdot(x-6)$

$\therefore V'(x) = 0 \iff x = 2$ or $x = 6$.

Furthermore,

$$V'(x) = 12x^2 - 96x + 144$$

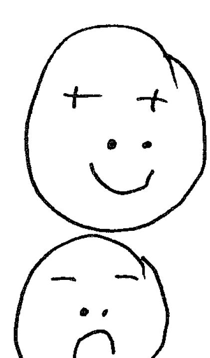
$$\Downarrow$$

$$V''(x) = 24x - 96 = 24(x-4)$$

So $\quad V''(x) = 0 \iff x = 4.$

Moreover, $\quad V''(x) > 0 \iff x > 4$ while

$V''(x) < 0 \iff x < 4$



So the graph of V is concave-up for $x > 4$ while the graph of V is concave down for $x < 4$.

$\therefore (4, V(4)) =$ an <u>inflection point</u> of the graph of V.

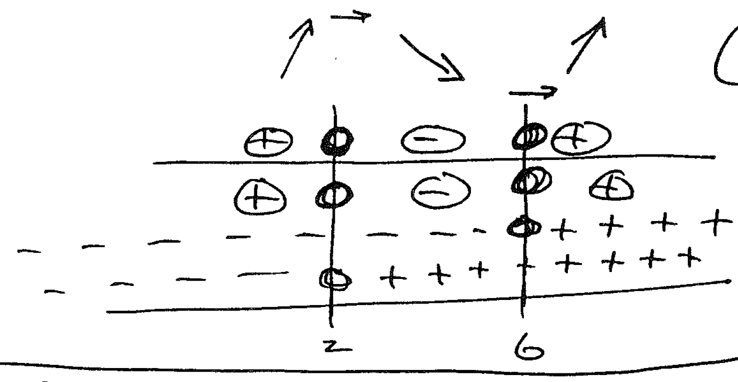
Behavior of V

$V'(x) = 12 \cdot (x-2) \cdot (x-6)$

$(x-2) \cdot (x-6)$

$x-6$

$x-2$

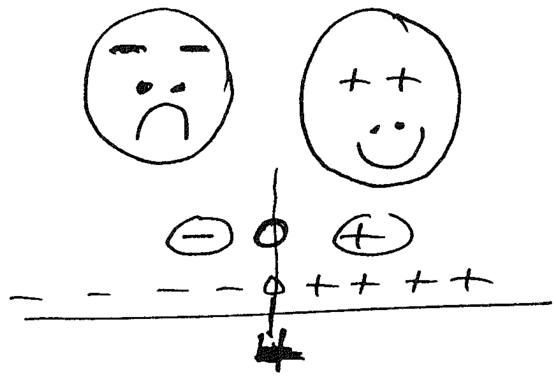


Sign Graph of $V'(x) = 12(x-2) \cdot (x-6)$

Behavior of V

$V''(x) = 24(x-4)$

$x-4$



Sign Graph of $V''(x) = 24(x-4)$

Behavior of V

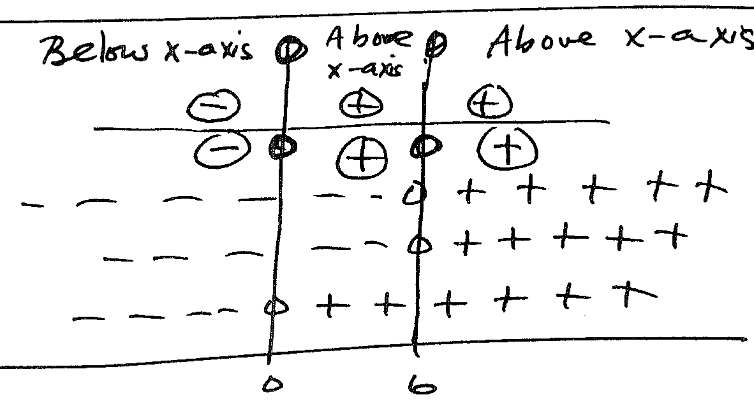
$V(x) = 4 \cdot (x-0)\ (x-6)^2$

$(x-0)\,(x-6)^2$

$x-6$

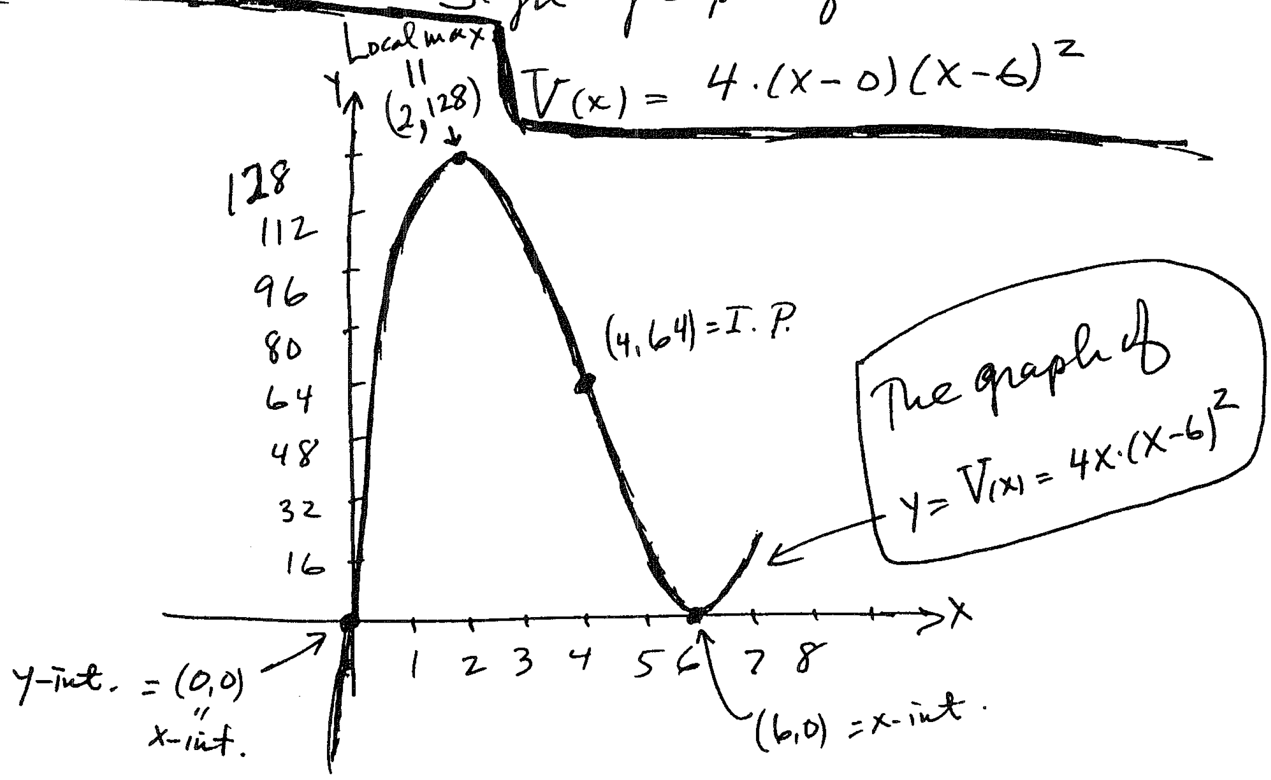
$x-6$

$x-0$



Sign Graph of

$V(x) = 4 \cdot (x-0)(x-6)^2$



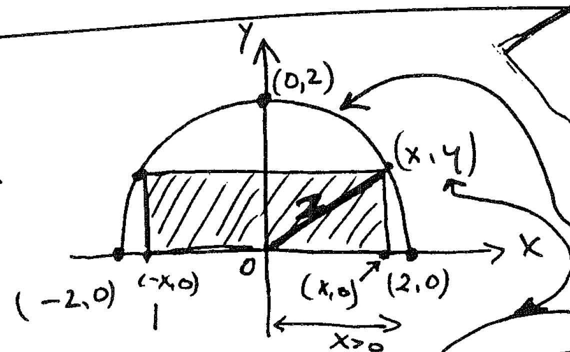
As you can see from the graph of $V(x)$, $x=2$ is the only point between $x=0$ & $x=6$ where $V(x)$ has a relative maximum (because it is the only point between $x=0$ & $x=6$ when $V'(x)=0$ & $V''(x)<0$). Globally, $V(x)$ itself has a local min. at $x=6$

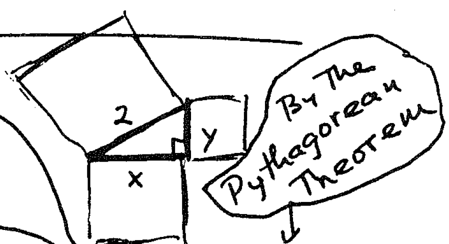
∴ for $0 \le x \le 6$, $V(x)$ has an absolute min. at $x=0$ & at $x=6$ (where $V(0)=V(6)=0$) & an absolute max. at $x=2$ where $V(2) = 4 \cdot 2 \cdot (2-6)^2 = 8 \cdot (-4)^2 = 8 \cdot 16 = 128$.

∴ The maximum volume is 128 cubic inches & each cut-out square should be 2 inches long per side!

$\boxed{\text{Problem 2}}$

$\boxed{\text{Step 1}}$: →

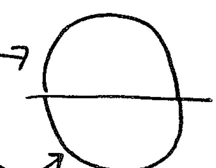




By The Pythagorean Theorem

$x^2+y^2=2^2=4$

$y=+\sqrt{4-x^2}$



$y=-\sqrt{4-x^2}$

$L :=$ Length $= 2x$

$H :=$ Height $= y = +\sqrt{4-x^2}$

$A :=$ Area $= L \cdot H = 2x \cdot \sqrt{4-x^2}$

Step 2 ∴ $A(x) = 2x \cdot \sqrt{4-x^2} = 2x \cdot (4-x^2)^{1/2}$, $\boxed{0 \le x \le 2}$

$\boxed{\text{In fact}}$ $\boxed{0 < x < 2}$

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$$A(x) = 2x\cdot(4-x^2)^{1/2} = 2\cdot\left[x\cdot(4-x^2)^{1/2}\right]$$

$$A'(x) = 2\cdot\left[x\cdot\left(\frac{1}{2}(4-x^2)^{-\frac{1}{2}}\cdot(0-2x)\right) + (4-x^2)^{1/2}\cdot 1\right]$$

$$= 2\cdot\left[\frac{x\cdot(-2x)}{2\sqrt{4-x^2}} + \sqrt{4-x^2}\right]$$

$$= 2\cdot\left[\frac{-\cancel{2}x^2}{\cancel{2}\sqrt{4-x^2}} + \sqrt{4-x^2}\cdot\frac{\sqrt{4-x^2}}{\sqrt{4-x^2}}\right]$$

$$= 2\cdot\left[\frac{-x^2+4-x^2}{\sqrt{4-x^2}}\right] = 2\cdot\left[\frac{4-2x^2}{\sqrt{4-x^2}}\right]$$

$$= 2\cdot 2\left[\frac{2-x^2}{\sqrt{4-x^2}}\right] = \frac{-4\,(x^2-2)}{\sqrt{4-x^2}}$$

$$= \frac{-4\cdot(x+\sqrt{2})\cdot(x-\sqrt{2})}{\sqrt{4-x^2}}$$

$$= \frac{-4\cdot\left[x-(-\sqrt{2})\right]\cdot\left[x-\sqrt{2}\right]}{\sqrt{4-x^2}}$$

Now $A(x)$ is defined $\forall\, x \in \mathbb{R} \ni$. $0 < x < 2$ & for all those x $A'(x)$ exists.

Now $A'(x) = 0 \iff x = -\sqrt{2}$ or $x = \sqrt{2}$

of these 2 values $(-\sqrt{2}$ & $+\sqrt{2})$ only $\sqrt{2}$ lies in the domain of A

Since

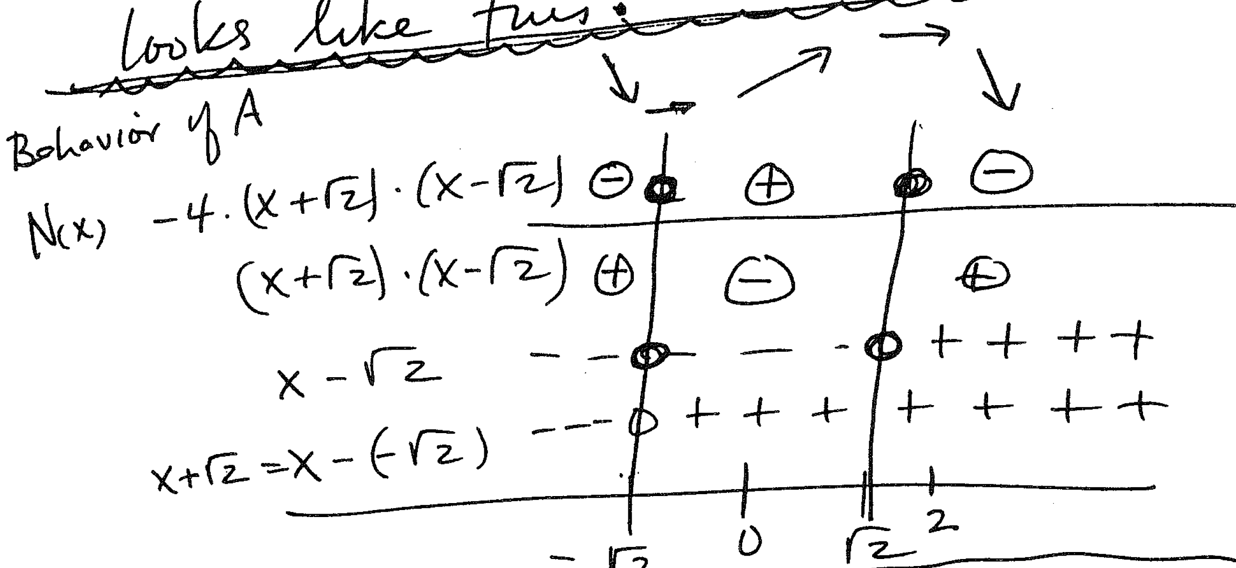
$$A'(x) = \frac{-4\,[x-(-\sqrt{2})]\cdot[x-\sqrt{2}]}{\sqrt{4-x^2}}$$

& Since $\sqrt{4-x^2} > 0$ whenever $0<x<2$,

the sign of $A'(x)$ coincides with that of its numerator

$$N(x) := -4\cdot[x-(-\sqrt{2})]\cdot[x-\sqrt{2}]$$

Now the sign graph for $N(x)$ looks like this:



$\boxed{\text{Signgraph of } N(x) = -4(x+\sqrt{2})(x-\sqrt{2})}$

This proves that $A(x)$ has a local maximum when $x=\sqrt{2}$. In that case

$$A(\sqrt{2}) = 2\cdot\sqrt{2}\,\left(4-(\sqrt{2})^2\right)^{1/2}$$

$$= 2\sqrt{2}\,(4-2)^{1/2} = 2\sqrt{2}\cdot\sqrt{2} = 2\cdot 2 = 4$$

<u>Result</u>: The rectangle has a maximum area of 4 square units & this occurs when the rectangle is $y = x = \sqrt{2}$ units <u>high</u> and $2x = 2\sqrt{2}$ units <u>long</u>.

<u>Note</u>: Here there is no need to find $A''(x)$.

Clearly $A'(x) = -4\left(\dfrac{x^2-2}{\sqrt{4-x^2}}\right) \Rightarrow A''(x) =$ a mess.

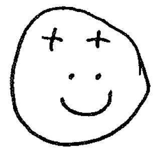
Yet, if you have courage you can verify that (about 9 steps later)

$$A''(x) = \frac{4x \cdot (x^2-6)}{(4-x^2)\sqrt{4-x^2}}$$

so from the associated sign graph

$A''(x) < 0$ if $0 < x < 2$

while $A''(x) > 0$ if $2 < x < \sqrt{6}$



This again confirms for us that $(\sqrt{2}, A\sqrt{2}) = (2, 4)$ = a maximum for A on $(0, 2)$.

$\boxed{\text{Problem 3}}$ Let x = one number &
y = the other number.

Then $x > 0$ & $y > 0$ & $\boxed{x + y = 20.}$

$\Downarrow$

The product of x & y is $\boxed{y = 20 - x}$

$$P = x \cdot y \stackrel{\downarrow}{=} x \cdot (20 - x).$$

$$\therefore \quad P(x) = x \cdot (20 - x) = 20x - x^2, \quad 0 < x < 20.$$

Then

$$P(x) = 20x - x^2$$

$$\Downarrow$$

$$P'(x) = 20 - 2x = 2\cdot(10-x)$$

$$= 2\cdot(-1)\cdot(x-10)$$

$$= -2\,(x-10)$$

$$\Downarrow$$

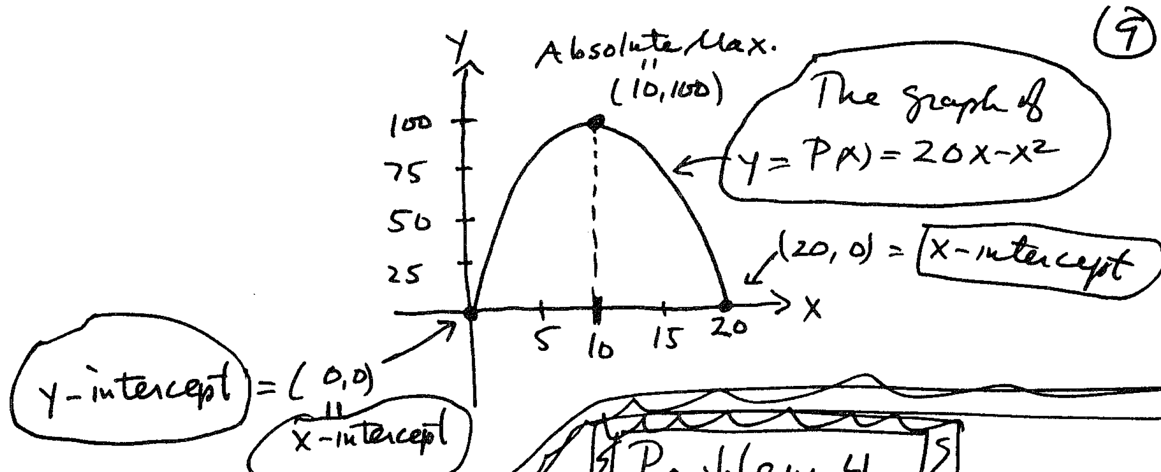
$$P''(x) = -2\cdot(1-0) = -2 < 0$$

Sign graph of $P'(x)$:

Behavior of P	$\nearrow$	$\rightarrow$	$\searrow$
$P'(x) = -2\cdot(x-10)$	$+$	0	$-$
$x-10$	$- - -$	0	$+ + + +$
		10	

Thus, $P \uparrow$ for $x < 10$ while $P \downarrow$ for $x > 10$.
& the graph of P has a horizontal tangent at $(10, P(10))$. Moreover $P''(x) = -2 < 0$ $\forall x$ implies that the graph of P is everywhere concave down ☹ !!! Thus

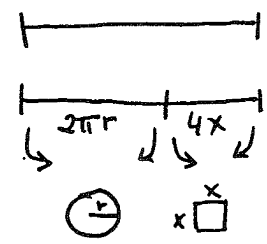
P has an absolute maximum when $x = 10$
In this case $y = 10$ & $P = x\cdot y = 100$.



Problem 4

$$A = \pi r^2 + x^2$$

$$L = 2\pi r + 4x = 4 \text{ feet}$$



A Constant

$$L = 2\pi r + 4x = 4 \text{ feet}$$

$$\Downarrow$$

$$0 = \frac{dL}{dr} = 2\pi + 4\frac{dx}{dr}$$

$$\Downarrow$$

$$\boxed{\frac{dx}{dr} = \frac{-2\pi}{4} = -\frac{\pi}{2}}$$

$$A = \pi r^2 + x^2, \quad 0 \le 2\pi r \le L = 4$$

$$\Downarrow$$

$$0 \le r \le \frac{L}{2\pi} = \frac{2}{\pi}$$

$$\frac{dA}{dr} = 2\pi r + 2x\frac{dx}{dr} = 2\pi r + 2x\cdot(\ \ )$$

$$= 2\pi r + 2x\cdot\left(-\frac{\pi}{2}\right)$$

$$= 2\pi r - \pi\cdot x = \pi\cdot(2r - x)$$

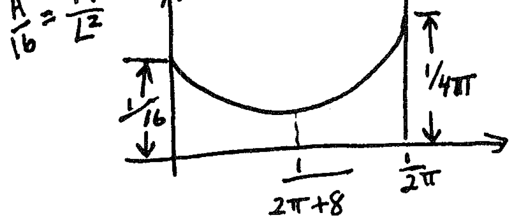
$$\therefore \frac{dA}{dr} = \pi\cdot(2r-x)$$

$$\Downarrow$$

$$\frac{d^2A}{dr^2} = \pi\cdot\left(2 - \frac{dx}{dr}\right) = \pi\cdot\left[2 - \left(-\frac{\pi}{2}\right)\right] = \pi\cdot\left(2 + \frac{\pi}{2}\right) > 0$$

$\therefore$ The graph of A as a function of r is everywhere concave-up, so A has an absolute minimum where $\frac{dA}{dr} = 0$, i.e., where $x = 2r$. Moreover, when $x = 2r$, then $4 = L = 2\pi r + 4x = 2\pi r + 4\cdot 2r = (2\pi + 8)\cdot r$

so $r = \frac{L}{2\pi+8} = \frac{4}{2\pi+8} = \frac{2}{\pi+4} \approx 0.28$ & $x = 2r = \frac{4}{\pi+4} \approx 0.56$



$0 \le r \le \frac{L}{2\pi} = \frac{2}{\pi}$

Now $r = 0 \Rightarrow x = \frac{L}{4} = 1$, & $A = x^2 = 1$

$r = \frac{2}{\pi} \Rightarrow x = 0$ & $A = \pi\cdot\frac{4}{\pi^2} = \frac{4}{\pi}$

At the minimum, $r = \frac{1}{2}\left(\frac{L}{\pi+4}\right)$, $x = \frac{L}{\pi+4}$, $A = \frac{L^2}{4\pi+16}$

Thus, don't cut the wire; use all for the circle

$\therefore$ there is no solution if one must cut the wire!!!

Math 2107

Max-Min Problems

1. An open-top box is to be made from a square piece of tin, 12 inches on a side, by cutting small equal squares from each corner of the 12-by-12 inch sheet of tin and turning up the sides. How large a square should be cut from each corner in order that the box shall have as large a volume as possible?

   $\boxed{\text{Answer}}$: Cut out a square of 2 inches per side for a max. vol. of 128 inches^3.

2. A rectangle is to be inscribed in a semi circle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

   $\boxed{\text{Answer}}$: The area has a max. value of 4 square units when the rectangle is $y = x = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long.

3. Find 2 positive numbers whose sum is 20 such that their product is as large as possible.

   $\boxed{\text{Answer}}$: 10 & 10.

4. Four feet of wire is to be cut into 2 pieces one of which is to be bent to form a circle & the other to form a square. How should the wire be cut if the sum of the areas enclosed by the 2 pieces is to be a maximum?